Multivariable Calculus with Understanding and How to Assess It

Daniela Velichová

Slovak University of Technology, Bratislava, Slovakia

Abstract

Some ideas are presented in this paper on how to use geometric interpretation for better understanding of selected problems related to multivariable calculus. Samples of a possible approach to assessing this understanding are given, and several examples of test problems used at the Slovak University of Technology in Bratislava are included.

Introduction

Multivariable calculus and the behaviour of functions of more variables is an advanced topic, extremely important at the technical universities, which can be understood properly with a certain effort of abstract thinking only. Presenting the concepts with a proper geometric interpretation might help a lot. Obviously, this is not an easy task. For instance, to achieve real students´ comprehension of the concept of a function of two variables and its behaviour, we must use some of the computer algebra systems to visualise its graph that is a surfaces in the three-dimensional space. Understanding of this representation requires a certain capability of 3D geometry and knowledge about the projection of the space to the plane. Problems are easy to appear as the graphs are often of rather peculiar and not easy-to-convey shapes. Particularly, special knowledge is tacitly assumed and required about tangent plane to the surface in an arbitrary point. This must be a surface regular point, and with respect to its type – elliptic, parabolic, or hyperbolic, the form of the tangential superposition rather differs. In a hyperbolic point the most "uncertain" situations can be expected.

Another problem can arouse with detecting extrema points of a function of two variables. If the tricky criterion based on the value of the zero Hesse determinant fails (the second derivative test), geometric interpretation and intuitive approach could be very helpful. But we can hardly expect our students, who most likely have not heard much about geometry in 3D and intrinsic properties of surfaces, to be able to sketch graphs of functions in two variables, and to solve related problems successfully. Therefore we usually simply assess the calculation skills and mastering of the mechanical usage of formulas learnt by heart without proper understanding and knowledge of their powerful meaning and decision power. The question is: Can we somehow overcome these unpleasant consequences of the recent development in maths education?

Elementary problems on functions of two variables

Multivariable calculus is a mathematical subject that appears within the study subject Mathematics II, and it is compulsory for all specialisations in the second semester of the bachelor study programmes at the Slovak University of Technology in Bratislava. The contents and difficulty level of the multivariable calculus course might differ for different specialisations. Students at the Faculty of Mechanical Engineering are supposed to cover and understand the theoretical backgrounds of the following topics: concept of the *n* dimensional Euclidean space and its basic properties, functions of *n* variables - domains of definition, graphs and limits, continuity, partial derivatives and total differentials. More advanced topics of local, constrained and global extrema are studied on practical examples for functions of 2 variables. Almost nobody among our students have heard before about 3D geometry and some of the projection methods that must be applied to illustrate simple graph of a function of 2 variables. To be able to sketch these graphs means therefore to explain students some of the basic principles of the orthographic projection, while there is no time and space reserved in the curricula for these topics. Generally, we simply use CAS and their commands to plot the graphs on practical exercises in computer laboratories during the semester, and we try to avoid this question at the examination tests, as these are not organised in laboratories with computers available. The variety of problems to be assessed is therefore very limited. Further problems dealing with function extrema are assessed mostly on the numerical basis, which means that students can learn by heart some formulas and calculate some data without proper understanding and ability to represent and apply these, for instance geometrically. A study has been carried on at the faculty about how students are able to solve certain problems on functions of 2 variables using system Mathematica.

For example, the problem to find the equation of the tangent plane to the graph of function $f(x, y) = y\cos(3x + 2y)$ at the tangent point $T = [0, \pi, ?]$ can be solved easily by simple calculation of function partial derivatives and their values in the given point from the well known formulas, while resulting equation of the tangent plane is $y = z$. From the illustration in Fig.1 it is clear that this plane is tangent to the surface not only at the point *T*, but the tangency superposition is more specific, as point *T* is a parabolic point of the surface. Tangent plane $y = z$ is tangent to the surface in all points on lines with equations $y - z = 0$, $3x + 2y = 2k\pi$, $k \in Z$. Students could not analyse this important property straightforwardly from the calculations themselves, but they were able to prove it by solving the system of two equations, namely $y = z$, $z = y\cos(3x + 2y)$, as they tried to find common points of the graph of function and the determined tangent plane in the given point.

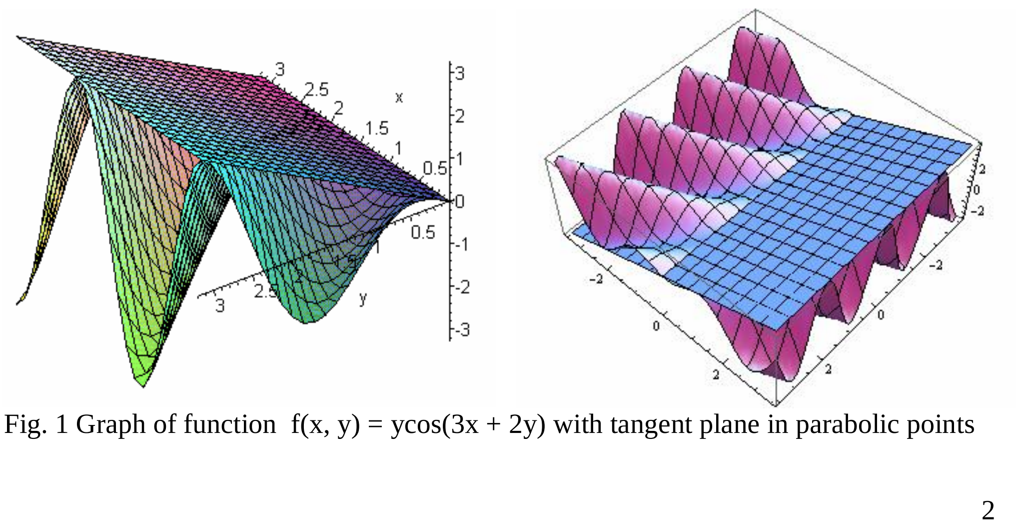

Fig. 1 Graph of function f(x, y) = ycos(3x + 2y) with tangent plane in parabolic points

Anther example isfinding the equation of the tangent plane to the graph of the function $f(x, y) = 1 - \sqrt[3]{x^2 + y^2}$ in the point T = [-2, 2, ?] leads to the plane determined by the equation $x - y + 3z - 1 = 0$. Tangent point T = [-2, 2, -1] is the surface hyperbolic point and the determined tangent plane intersects the surface – graph of the function, in planar curve represented by the implicit equation $x - y - 3\sqrt[3]{x^2 + y^2} + 2 = 0$.

An orthographic view of the tangent plane and surface configuration in the image plane $x + y = 0$ passing through the coordinate axis z clearly illustrates the situation, whereas tangent plane appears in the edge view, see fig. 3. Students are able to sketch the configuration, as from the condition $x = -y$ simply yields that equation of the surface and image plane intersection curve is $z = 1 - \sqrt[3]{2x^2}$ and the edge view of the tangent plane in the defined image plane is line determined by the equation $z = \frac{1 - 2x}{3}$.

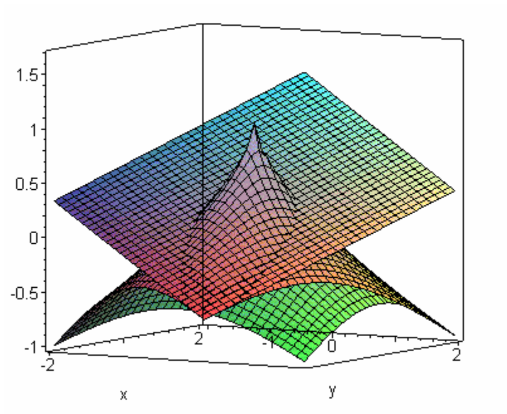

Fig. 2 Graph of function $f(x, y) = 1 - \sqrt[3]{x^2 + y^2}$ with tangent plane in hyperbolic point

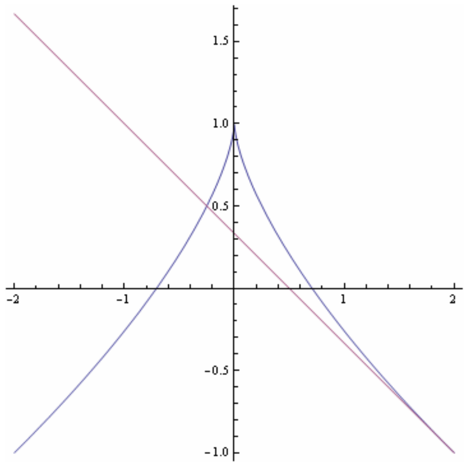

Fig. 3 Orthographic view of the tangent configuration

Local, constrained and global extrema of functions of two variables

Part of the assessment in the course Mathematics II is performed in the form of focused projects elaborated by students on specific topics. Complex problems about extrema of functions of two variables were solved as particular tasks delivered at the end of the testing period. Projects aimed to evaluate the understanding of the basic concepts as local, global and constrained extrema and their relation to the behaviour of the function of two variables. Students used the Mathematica system for illustrations and some calculations. Several examples of problems solved in the projects are presented in the following.

Problem 1.

Consider the functions a) $f(x, y) = \dfrac{3x}{x^2 + y^2 + 1}$, b) $f(x, y) = 4 - x^2 - 2xy - 3y^2$.

Using CAS sketch the graph of the function $f(x, y)$ of two variables on the domain

a) $D = \{(x, y) \in R^2 : -4 \le x, y \le 4\}$, b) $D = \{(x, y) \in R^2 : -2 \le x, y \le 2\}$.

Calculate the gradient of function $f(x, y)$, sketch the curves $f_x(x, y) = 0,\ f_y(x, y) = 0$, identify the existence of critical points of the function f, and determine these points exactly by solving the system of equations. Use the second derivative test to classify each critical point, and calculate the extremal values of the function f and equations of tangent planes to the graph of function in the points of the strict local extrema. Compare differences and similarities of the behaviour of the two functions in a) and b).

Critical points are the intersection points of the two sketched curves (hyperbola and two lines) located in coordinate plane xy, K_1 = [-1,0,$-\frac{3}{2}$] and K = [1,0,$\frac{3}{2}$], while the strict local minimum that is equal to $-\frac{3}{2}$ can be determined at the point [-1,0], and the strict local maximum equal to $\frac{3}{2}$ at the point [1,0]. Students use the second derivative test to prove this fact.

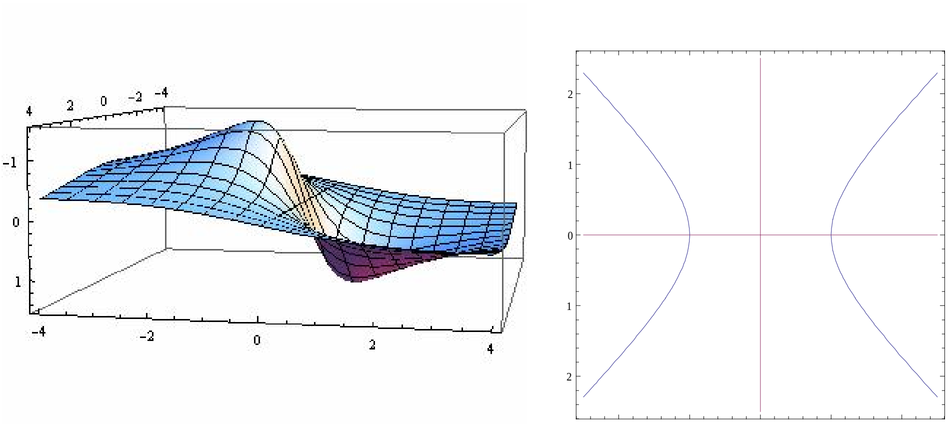

Fig. 4 Graph of function f and curves $f_x = 0$, $f_y = 0$ in a).

For the problem b) there exists a unique critical point [0, 0] determined as the intersection point of two lines, while the strict local maximum at this point is equal to 4.

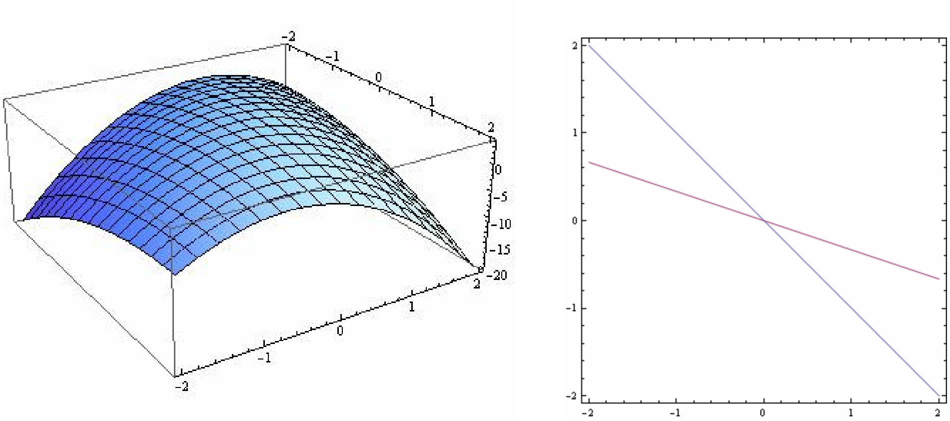

Fig. 5 Graph of function f and curves $f_x = 0$, $f_y = 0$ in b).

Students are rather successful in solving this sort of problem, and they understand the geometric representations of the analytic methods and algorithms behind them. Projects with problems related to constrained extrema of functions of two variables and global extrema of functions defined on closed sets in R2 were regarded as more difficult.

Problem 2.

Find global extrema of the function $f(x, y) = x^2 + y^2 - 12x + 16y$ on the set of points determined by the inequality $x^2 + y^2 \leq 25$. Solve the problem in the four steps:

a) Find local extrema inside the set M determined by the inequality (calculate function values at all critical interior points of set M).

b) Find the least and the greatest values of function f on the boundary of the set M (use the method of Lagrange multiplier).

c) Identify the greatest and the least of all the found values of function f in a) and b), which is the global maximum and the global minimum of function f.

d) Plot the graph of the function $f(x, y)$ and curve on this surface that is determined by the points of the circular boundary of the set M and characterize this curve geometrically.

Solution of the problem can be estimated after the plot required in the point d) is performed, and illustrated graphical information and be geometrically analysed (Fig. 6). To perform the plot, it is suitable to find parametric equations of surface that is the graph of the function f (paraboloid of revolution with equation $z = (x - 6)^2 + (y + 8)^2 - 100$, with axis in the coordinate axis z and vertex at the point [6, -8, -100]) and circular cylindrical surface with axis in the coordinate axis z and radius 5 intersecting this graph in the ellipse located in the plane determined by the equation $12x - 16y + z - 25 = 0$.

Solution $x = 6$, $y = -8$ of system of equations $2x - 12 = 0$, $2y + 16 = 0$ determines the critical point [6, -8], which is not the point inside the set $M = \{[x, y] \in R^2: x^2 + y^2 < 25\}$ and therefore there exist no local extrema of function f on the interior of the set M.

Using the method of Langrange multiplier *l* two critical points [-3, 4] and [3, -4] can be calculated for values *l* = -3, *l* = 1, where constrained maximal value 125 of the function *f*(*x*, *y*) is achieved in the first point and function constrained minimum -75 in the second point. Function global extrema on the set *M* coincide with the constrained extrema.

More examples and solved problems can be found in materials cited in the references below and in the on-line database of e-learning educational materials linked to the Central EVLM portal of the European project European Virtual Laboratory of Mathematics at the address www.evlm.stuba.sk.

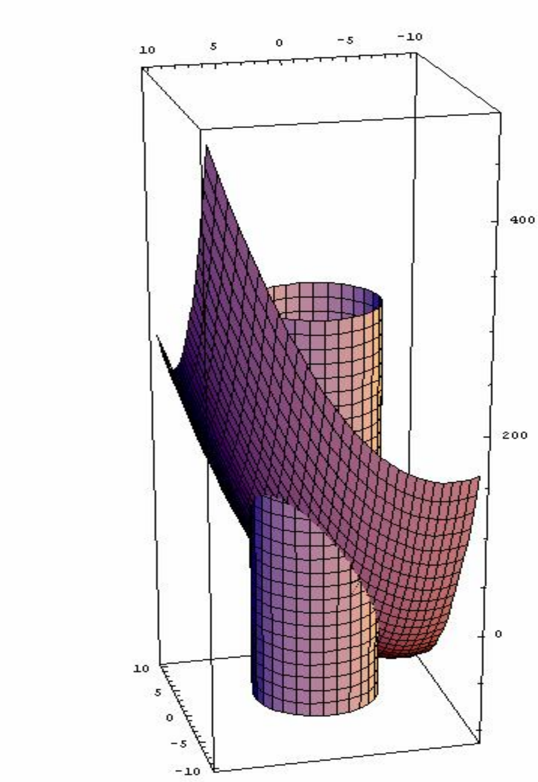

Fig. 6 Global (and constrained) extrema of function of two variables

References

Dobrakovová, J. (2006) *Mathematics II, Lectures in English.* Vydavateľstvo STU v Bratislave.

Rektorys, K. (1969) *Survey of Applicable mathematics.* Illife Books Limited, London.

Velichová, D. (2007) *Mathematics II, electronic textbook.* http://evlm.stuba.sk/~velichova/Matematika2/Kniha/kniha.html (in Slovak) (7 March 2008).

Formatiert: Schriftart: Kursiv