Qualitative methods for nonlinear differential equations – the Solow model of long time economic growth

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Abstract

Many basic mathematical subjects and procedures of great relevance for engineering practice are not included in mathematics curriculum of universities. Utilizing computers actually requires more “higher mathematics” knowledge than was necessary in the past. This fact will be demonstrated for nonlinear differential equations. Simple qualitative methods to determine the essential solution behavior of nonlinear equations are given in the paper. This will be applied to the Solow model.

Introduction

Over recent years the mathematical starting knowledge of students has decreased to a very poor standard which persists at the moment (Berger, M. and Schwenk, A. 2006, Brüning, H. 2004). It is the most important problem to overcome to ensure high mathematical standards in teaching process. (Schott, D. 2007). Future engineers need computers for calculation but there is no time to develop maths skills and the sensible use of computers. Indeed one cannot trust numerical computer results and there is a need for advanced mathematical methods for verifying the results of computing. In the following some qualitative methods for linear and nonlinear autonomous differential equations are explained briefly and applied to the Solow model. Further the stability in the sense of Lyapunov of equilibrium solutions is considered. This paper is a proposal to adapt the methods under consideration in math lectures for engineers. Until now they are not included in maths for European engineers (Mustoe, Lawson 2002).

Autonomous Differential Equations

An autonomous system of ordinary differential equations has the form:

\[ \frac{dx}{dt} = F(x). \]  

(1)

For every \( x \in \mathbb{R}^n \) a velocity \( F(x) \in \mathbb{R}^n \) is given. From the vector \( F(x) \) we know how \( x \) changes. Existence and Uniqueness of the solution \( x_1(t), x_2(t), ..., x_n(t) \) can be proved by requiring the function \( F \) to be Lipschitz continuous. By uniform boundedness of the Jacobian

\[ \frac{\partial F}{\partial x} \]

Lipschitz continuity of \( F \) is ensured. There is a maximal interval \( I \) where the solution is defined. We restrict ourselves to planar systems of equations having two state variables (\( n = 2 \)). Each system state corresponds to a point of the \((x_1, x_2)\) – plane called phase plane. If \( t \) varies over the maximal interval \( I \) the curve \((x_1(t), x_2(t))\) is called a trajectory, path or orbit. The set of all trajectories is called phase portrait. Every trajectory is uniquely defined
by an initial value \( x_0 = x(t_0) = (x_1(t_0), x_2(t_0)) \). Different trajectories will not intersect at any time. An equilibrium or fixed point \( x_e \) of an autonomous system is a state with zero velocity. Equivalently, every zero of \( F \) is a fixed point, i.e. \( F(x_e) = 0 \). The last equation is the stationary condition for the dynamics of the system. Equilibrium points may be stable or unstable. More generally, a solution \( x(t) \) to the differential equation is called stable in the sense of Ljapunov if for all \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for every solution \( y(t) \) the implication holds
\[
||x(t_0) - y(t_0)|| < \delta \rightarrow ||x(t) - y(t)|| < \varepsilon
\]
for all \( t > 0 \). It is called unstable otherwise. If \( x(t) = x_e \) is an equilibrium point one says that the fixed point is stable or unstable. A solution is asymptotically stable if it is stable and furthermore \( y(t) \to x(t) \) as \( t \to \infty \). An asymptotically stable solution represents the long time behavior of nearby solutions of the dynamical system. In this way stability is a local property. If all solutions converge to \( x(t) \) as \( t \to \infty \) independently of the initial values then \( x(t) \) is a globally stable solution. An asymptotically stable fixed point \( x_e \) is called a sink. An unstable equilibrium point is a source if all trajectories run away from the equilibrium point or a saddle if some trajectories move to \( x_e \) and some lead away from it. If a fixed point is stable but not asymptotically stable than it is called a center. We can classify fixed points for linear systems given by the equation
\[
\frac{dx}{dt} = Ax.
\]
Here \( A \) is a \((2x2)\)-matrix with trace \( \tau \) and determinant \( \Delta \). For every \( A \) the solution \( x(t) = 0 \) is a fixed point. There may exist more than one equilibrium point \( (\Delta = 0) \). Linear systems are fully characterized by the eigenvalues and eigenvectors of \( A \). From \( \Delta \neq 0 \) it follows that \( 0 \) is a unique equilibrium point. Suppose that \( A \) has two real eigenvalues \( \lambda_1 < \lambda_2 \). Let \( \Delta \neq 0 \), there are three cases to consider

- Saddle: \( \lambda_1 < 0 < \lambda_2 \);
- Sink: \( \lambda_1 < \lambda_2 < 0 \);
- Source: \( 0 < \lambda_1 < \lambda_2 \).

The saddle and the source are unstable, the sink is globally stable. For the saddle the eigendirection of \( \lambda_1 \) (\( \lambda_2 \)) is a so-called stable (unstable) manifold. The remaining cases
(complex, multiple, zero eigenvalues) can be found in detail in Hirsch, Smale and Devaney (2004). Obviously a fixed point $x_e = 0$ is asymptotically stable if all eigenvalues have negative real part. Contrary the fixed point is unstable if one eigenvalue has positive real part.

For our purposes the Hartman-Grobman theorem for the so called hyperbolic case where all eigenvalues of the Jacobian have nonzero real part suffices for the nonlinear case too (Olver 2006).

**Theorem** Let $x_e$ be an equilibrium point for the first order ordinary differential equation \( \frac{dx}{dt} = F(x) \). If all eigenvalues of the Jacobian matrix $\frac{\partial F}{\partial x}(x_e)$ have negative real part then $x_e$ is asymptotically stable. On the other hand, if the Jacobian $\frac{\partial F}{\partial x}(x_e)$ has one (or more) eigenvalues with positive real part, then $x_e$ is an unstable equilibrium point.

If the Jacobian at a fixed point has eigenvalues with zero real part the theorem provides no insight for the stability. In fact, in this case the fixed point can be stable or unstable. An answer can be found by Lyapunov’s methods, in particular a Lyapunov function $V(x): \mathbb{R}^n \to \mathbb{R}$ is required.

**Definition** A Lyapunov function for the first order autonomous system (1) is a continuous function $V(x)$ that is non-increasing on all solutions $x(t)$, meaning that $V(x(t)) \leq V(x(t_0))$ for all $t > t_0$. A strict Lyapunov function satisfies the strict inequality $V(x(t)) < V(x(t_0))$ for all $t > t_0$, whenever $x(t)$ is an non-equilibrium solution. If $V$ is continuously differentiable it is a (strict) Lyapunov function if and only if it satisfies the inequality

\[
\frac{d}{dt} V(x(t)) = \nabla V(x) \cdot F(x)(\leq) \leq 0. \tag{2}
\]

**Theorem** Let $V(x)$ be a (strict) Lyapunov function for the system (1). If $x_e$ is a strict local minimum of $V$, then $x_e$ is a (asymptotically stable) stable equilibrium point. On the contrary, any critical point of a strict Lyapunov function which is not a local minimum is an unstable equilibrium point.

For general nonlinear systems it can be difficult to find a Lyapunov function. The Krasovskii-Lasalle theorem gives less restrictive conditions, sufficient for an equilibrium point being asymptotically stable.

In systems theory stationary points and periodic solutions are the most important characteristics of a dynamic system. In the case of planar autonomous systems the theorem of Pioncaré-Bendixson gives sufficient conditions for the existence of closed orbits (Hirsch, M.W., Smale, S. and Devaney, R. L. 2004). The theorem of Bendixson-Dulac gives a condition implying that no closed orbit exists. In the next section the theory is applied to an economical application.

**The Solow-Model**

We consider the production function $F$ for one final good $Y=F(K,AL)$. Here $Y$, $A$, $K$, $L$ denote output, labour productivity, capital, and labour. All these variables are functions of time. The function $F$ satisfies some properties:

- $F \in C^1$, $F$ is strictly increasing and strictly concave
- $F$ is an CRS function
The latter means if the inputs of $F$ are changed by a certain proportion, then the output also changes by the same proportion (e.g. $F$ is homogeneous of degree one). A standard example is the Cobb-Douglas function

$$F(K,AL) = K^\alpha (AL)^{1-\alpha}, \ 0<\alpha < 1.$$ 

Solow assumed that $A(t) = A(0)e^{rt}$ and $L = L(0) e^{nt}$ with exogenous rates of growth $\gamma$ and $n$. For given $K(0)$ the accumulation of capital is

$$\dot{K} = I - \delta K$$

The investment $I$ is assumed to be equal to the savings $sY$ a constant fraction of output $Y$. Accordingly, the capital accumulation equation reads

$$\dot{K} = sY - \delta K = sF(K,AL) - \delta K = AL sF(AL), I - \delta K.$$ 

Defining $k = K/(AL)$ and the intensive production function $f(k) = F(K/(AL), I)$ one arrives at

$$\frac{\dot{K}}{K} k = \frac{\dot{K}}{AL} = sf(k) - \delta K.$$ 

By logarithmic differentiation we find

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - (n + \gamma).$$

Finally the fundamental equation of growth is

$$\dot{k} = sf(k) - (n + \gamma + \delta)k.$$ 

A fixed point $k_e$ is given as solution of

$$sf(k_e) = (n + \gamma + \delta)k_e.$$ 

For the Cobb-Douglas production function $f(k) = k^\alpha$ one computes $k_e = \left(\frac{s}{n+\gamma+\delta}\right)^{1-\alpha}$. 

We set $G(k) = sk^\alpha - (n + \gamma + \delta)k$. Assuming $0<\alpha<1$ we find $G'(k_e) < 0$ and deduce that $k_e$ is an asymptotically stable equilibrium point. Obviously the nontrivial steady state $k_e$ is unique. Furthermore it can be shown using the Inada conditions ($f(0) = 0, f'(0) = \infty, f'(\infty) = 0$), for more general $f(k)$, that there is also an unique nontrivial fixed point which is globally stable.

Moreover in the Cobb-Douglas case $f(k) = k^\alpha$ the fundamental equation of growth is a Bernoulli equation

$$\dot{k} = sk^\alpha - (n + \gamma + \delta)k.$$ 

It has a closed solution easily computable by hand. Formally a strict Lyapunov function for this one-dimensional equation is

$$V(k) = -\frac{s}{1+\alpha} k^{\alpha+1} + \frac{n+\gamma+\delta}{2} k^2.$$
This can be deduced from (2) by
\[ V'(k)k = -(sk^a - (n + \gamma + \delta)k)^2 < 0. \]

Thereby we conclude that the fixed point is asymptotically stable and also globally stable.

**The Solow-Model with logistic manpower**

The exponential growth \( L = L(0)e^{nt} \) of Labour is unrealistic. It is replaced by logistical growth. Now for \( L > 0 \) one has
\[ \frac{L}{L} = a - bL, \ a > 0, \ b > 0. \]

This modification seems to be marginal but it makes the model much more uncomfortable. If we use the Cobb-Douglas production function as before we find a (decoupled) system of differential equations:
\[
\begin{align*}
\dot{k} &= sk^a - (a - bL + \gamma + \delta)k \\
\dot{L} &= (a - bL)L.
\end{align*}
\]

If \( k(0) \) and \( L(0) \) are known there exists an unique solution of the system. Furthermore the fixed point is
\[ (k_e, L_e) = \left( \left( \frac{s}{\gamma + \delta} \right)^{\frac{1}{1-a}}, \frac{a}{b} \right), \]

The Jacobian at the fixed point reads
\[ J(k_e, L_e) = \begin{pmatrix} -(1 - \alpha)(\gamma + \delta) & bk_e \\ 0 & -a \end{pmatrix}. \]

Because the eigenvalues \(- (1 - \alpha)(\gamma + \delta)\) and \(-a\) are both negative the fixed point is an asymptotically stable sink. In this model there exists no closed orbit. Proving this we multiply \( \dot{k} \) and \( \dot{L} \) by \( \frac{1}{kL} \) and differentiate
\[
\nabla \cdot \left( \frac{\dot{k}}{\frac{1}{kL}} \right) = \nabla \cdot \left( \frac{sk^{a-1} - (a - bL + \gamma + \delta)}{(a-bL)} \right) = s(1 - \alpha) \frac{k^{a-1}}{L} \frac{b}{k} < 0.
\]

From the theorem of Bendixson-Dulac it now follows that there doesn’t exist periodic solutions.

The solution of the logistic equation \( \dot{L} = (a - bL)L, \ L(0) = L_0 \) is given as \( (t) = \frac{aL_0a^{at}}{a + bL_0(e^{at} - 1)}. \) The growth rate \( n(t) \) of \( L \) is \( n(t) = \frac{\dot{L}}{L} = \frac{a(a - bL)L_0}{a + bL_0(e^{at} - 1)}. \)

It can be inserted instead of \( (a - bL) \) into the first equation of the system (3), (4). The second equation is solved and the system reduces to an equation which is not autonomous
\[ \dot{k} = sk^a - (n(t) + \gamma + \delta)k. \]

The solution of this equation can be given using Hypergeometric functions (Brida 2005).
References


