Modelling by Differential Equations – from Properties of Phenomenon to its Investigation

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Abstract

The Panevezys campus of Kaunas University of Technology comprises the Business and Administration Faculty and the Faculty of Technology. They host students of business administration, civil engineering, electrical engineering, management and mechanical technology. Students of all specialties need mathematical knowledge for the solution of practical problems and for research from specific domains, such as mechanics, control, techniques, physics, economics etc.

The paper discusses methodological issues of teaching ordinary differential equations to the bachelor and master students from the Faculty of Technology and bachelor students from the Business and Administration Faculty.

Our experience of many years shows that students understand the theory of ordinary differential equations better if it is illustrated with applications. For building up models we use mechanical and economical problems and solve the inverse problems for ordinary differential equations as well. Our opinion is that inverse problems promote better understanding of the idea of modelling idea.

Introduction

The Panevėžys campus of Kaunas University of Technology consists of the Faculty of Technology and the Business and Administration Faculty. General higher mathematics topics are covered during two terms, 3 hours for theory and 2 or 3 hours for tutorials per week. It is possible to introduce first and second order ordinary differential equations (ODE). The level of proficiency of different mathematical topics slightly differs among different specialities. Engineers of all specialties will need mathematical knowledge for the solution of practical problems and for research in the fields of mechanics, control, techniques, and physics. Students of business and economics should master skills that will be useful in economics, statistics and econometrics. For students of both specialities we point out that differential equations are tools for solving real-world problems that have been discussed in lectures of other subjects.

During tutorials we firstly master finding solutions of particular ODEs. After that it is easier to draw attention to the applications of the ODE. Generally, modelling provides a chance to develop practical skills of applying mathematics in particular domains (Gershenfeld, 1998).
To build a model, student must make many decisions. Therefore it is convenient to divide the whole modelling process into separate phases (Shier & Wallenius, 1999).

We usually follow five steps when demonstrating investigations of applied problems by means of differential equations.

1. Analysis of the real-world problem. Usually it is a problem that students have already considered in engineering or economics lectures. This involves evaluation of the major known properties of phenomenon and the formulation of what properties should be investigated by modelling using an ODE as a tool.

2. Set up a mathematical model, i.e. write down a differential equation of the model. Our opinion is that in this step it is important to point out how the differential equation describes the known properties discussed in the first step and what fundamental laws of nature and economics should be taken into account (Самарский, 2001). Attention should be drawn to the fact that the ODE is an expression that includes first or higher order derivatives. Therefore it represents the rate of change of the properties of the system or phenomenon under discussion. This is a good point to review the mechanical and economic meaning of the first and second order derivatives.

3. Solve the ODE. Here one can often but not always focus solely mathematical knowledge.

4. Analysis of the solution of the ODE. This step generally remains mathematically oriented and it includes analysis of the solution’s closed-form expression if it is available. The analysis of the solution may include a discussion of the expression, evaluation its analytical properties, pointing out how the solution depends on boundary or initial values, plotting the graph etc.

5. Reformulate the mathematically discovered characteristics of the solution into engineering or economical language. This is a good place to compare the new properties and knowledge about the problem revealed by modelling with already known facts, to predict the behavior of the device or economical process and to check if it is what the engineers expected (Dym, 2004). This step also contributes to the validation of the model.

Practical examples

For example we consider the determination of the expression for demand for some good, depending on income, when the elasticity is constant.

1. Analysis of the problem. This problem comes from the microeconomics. Constant elasticity functions are often discussed in economic theory and they are considered as simple functions. But the analytical expression is not always given. So it is interesting to find this expression and to investigate properties of it from the mathematical viewpoint.

2. The definition of elasticity of demand (D) with respect to income \( x \) is \( D'x/D \) and the requirement to be constant yields \( (D'x/D)' = 0 \).
3. This is a second order ODE and there are no difficulties in showing its solution is \( D = c_2 x^3 \) with arbitrary constants \( c_1 \) and \( c_2 \).

4. When performing the mathematical analysis of the function \( D \), it is essential to point out that \( (D'x/D) = c_1 \). Therefore graphs of the demand function \( D \) with various elasticity \( c_1 \) values including \( c_1 > 1 \) and \( c_1 < 1 \) should be plotted.

5. The main result is that the demand of constant elasticity is a power function. The demand \( D \) has neutral elasticity, i.e., when \( c_1 = 1 \), only when \( D = c_2 x \) and in this case the demand is proportional to income \( x \). But the linear function of general form \( y = kx + b \) can not have the constant elasticity property.

Instructors of applied mathematics are typically more concerned with direct problems. In real life, the problem of interest to engineers and scientists is often an inverse problem. For instance, in many practical situations it is necessary to find coefficients of the ODE, parameters of the source term or parameters of the boundary values. The engineer seeks to determine these unknown parameters from collected measurements or other accessible knowledge about the solution of the given ODE.

Our experience shows that students understand the theory of ODE better if this theory is also illustrated with inverse ODE problems. In addition, considering inverse problems also promotes better comprehension of the essence of ODE models. The theory of inverse ODE problems nowadays is well developed, and has wide applications due to the accessibility of powerful computers.

When the educator introduces such problems, the major difficulty that students face is that of cause and effect, i.e. the direct problem and the inverse problem switch places with one another. Therefore the educator must pay special attention to this peculiarity and explain it.

As an example, we consider one problem that reveals methodological aspects of solution of the inverse problem.

We start from the boundary value problem that describes the transversal deflection \( u(x) \) of the heterogeneous beam when it is supported at points \( x = 0 \) and \( x = L \).

\[
\begin{align*}
\frac{d^2 M}{dx^2} &= F(x), \\
M(0) &= M(L) = 0, \\
D(x) \frac{d^2 u}{dx^2} &= M(x), \\
u(0) &= u(L) = 0.
\end{align*}
\]

where \( F(x) \) is a distributed load, \( M(x) \) is bending moment acting in the beam cross section, \( D(x) = E(x)I(x) \) is flexural rigidity of beam, \( E(x) \) is Young’s modulus of the beam material, \( I(x) \) is the second moment of the cross section area.

The first two equations of (1) have analytical solution
\[ M(x) = \int_0^x \int_0^x F(t) dt dv - x \int_0^x F(t) dt dv, \quad (2) \]

and provided that \( D(x) > 0 \), we obtain the solution of problem (1)

\[ u(x) = \int_0^x \int_0^x \frac{M(t)}{D(t)} dt dv - x \int_0^x \frac{M(t)}{D(t)} dt dv. \quad (3) \]

Hence we have reduced the problem (1) into boundary-value problem for the linear non-homogeneous differential equation

\[ \frac{d^2 u}{dx^2} = P(x), \quad u(0) = u(L) = 0 \quad (4) \]

where \( P(x) = M(x)/D(x) \). Assuming \( u_1(x) \) and \( u_2(x) \) are linearly independent solutions of the homogeneous problem. Then the Green’s function of the problem (4) is given by expression

\[ G(x,s) = \frac{u_1(x)u_2(s)}{W(s)} \left[ 1 - H(x-s) \right] + \frac{u_1(s)u_2(x)}{W(s)} H(x-s) \quad (5) \]

where \( W(s) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \) is the Wronskian and \( H(x) \) is Heaviside’s step function.

Then the solution of the problem (1) is

\[ u(x) = \int_0^L G(x,s)P(s)ds \quad (6) \]

Now we shall solve the inverse problem, i.e. we seek \( P(x) \) for given \( u(x), F(x), x \in (0, L) \). Notice that function \( M(x) \) becomes known when the problem (2) is solved. Hence one can derive the bending stiffness

\[ D(x) = \frac{M(x)}{P(x)} \quad (7) \]

and then, assuming that the second moment of the cross section area \( I(x) \) is known, one can also derive Young’s modulus distribution

\[ E(x) = \frac{D(x)}{I(x)}. \quad (8) \]
The problem (6) can be solved by applying numerical quadrature formula. The interval \((0, L)\) is divided by points \(s_i = iL/N, i = 0, N\) into \(N\) equal subranges \(S_i = (s_i, s_{i+1})\), \(i = 0, N - 1\) and then

\[
\int_0^L G(x, s)p(s)ds = \sum_{i=0}^{N-1} \int_{s_i}^{s_{i+1}} G(x, s)p(s)ds \approx \sum_{i=0}^{N-1} P\left(\overline{s}_i\right) \int_{s_i}^{s_{i+1}} G(\overline{s}_j, s)ds ,
\]

where \(\overline{s}_i = (s_i + s_{i+1})/2\) are middle points of intervals \(S_i\). If \(u_i = u(\overline{s}_i)\) are measurements then the solution of the inverse problem (6) is the solution of linear system

\[
AP = U ,
\]

where

\[
A_{ij} = \int_{s_i}^{s_{j+1}} G(\overline{s}_j, s)ds , \quad U^T = (u_1, u_2, \ldots, u_N) , \quad P^T = (p_1, p_2, \ldots, p_N) , \quad p_i = p(\overline{s}_i) .
\]

Now the educator can consider quite important real-world engineering problems. Beams made of materials like concrete, show degradation of flexural stiffness during their service life due to mechanical and environmental loadings. Using the method outlined above, we can investigate how the material parameters (Young’s modulus and flexural stiffness) change.

Finally, we note that, from the expression of the ODE it is sometimes possible to draw conclusions about which real-world problems and systems this ODE can serve as a model. For instance, when covering the first order differential equations, we consider the equation \(y' = k_1y - k_2y\). Educators can draw attention that the equation models general process changes that are influenced by two factors corresponding to terms \(k_1y\) and \(k_2y\). The first term increases the changes while the second contributes to the decrease.

**Conclusions**

1. Modelling skills should be taught by practically building models in concrete domains including data acquisition from these domains. Models become more meaningful when students collect their own data.
2. The background of successful modelling with differential equations is the knowledge of the meaning of derivatives.
3. The consideration of inverse ODE problems contributes to better understanding of the essence of modelling by differential equations.
4. When a particular differential equation is given it may be useful to discuss what processes and what relationships between properties of the process it can describe.

5. Modelling teaches students to apply mathematical concepts to solve real problems.

References


