Students’ Conceptions of Nothingness and their Implications for a Competency driven Approach to Curriculum

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Abstract

Competency has been the buzzword of higher education for at least a decade. The reasonable approach to describe what students should be able to do after completing coursework, however, falls short of the fact that mathematics, like any subject matter, contains inherent difficulties for students. Students usually need assistance in overcoming such difficulties. Competency driven approaches to curriculum tend to ignore this issue. Here, this problematic issue will be exemplified by investigating students’ difficulties with the concept of empty set on one hand and SEFI’s framework for a mathematics curriculum on the other hand.

Introduction

Students do not enter the learning process as empty vessels. What they bring with them in terms of preconceptions and prior beliefs significantly influences their learning. Competency driven approaches to teaching tend to underemphasise this issue by following a deductive approach: define a rather abstract concept to start with (competency, in this case) and deduce what that means for student learning.

Research needs to complement this arguably reasonable method with an inductive approach: observe students’ difficulties with subject matter and induce what that means for teaching. To support this argument I will present findings of an on-going research project focusing on students’ understanding of the concepts of empty set and empty word (the latter one being of paramount importance in automata theory). While both concepts in a way are an “incarnation of nothingness” they are unrelated to a large extent. Yet the data collected so far in exams, formative assessments, and student interviews strongly suggest that many students enter (and leave) their learning process with an incorrect understanding of the empty set. Furthermore they use this understanding of the empty set to construct their own understanding of the empty word.

Empty word and empty set are but two concepts in the vast “concept space” of mathematics. The SEFI framework for a mathematics curriculum (SEFI MWG, 2010) views such concepts as a key in defining mathematical competence as “the ability to recognize, use and apply mathematical concepts in relevant contexts and situations” (p. 3). In fact, the document specifically mentions the ability to “understand the concepts of a set, a subset and the empty set” (p. 28) as one of many very fundamental content-related mathematical competencies. But what does it mean to understand the concept of the empty set?

Basically, competency in general and understanding in particular are ill-defined concepts in the sense that scientific concepts need to be defined operationally. But what are the operations that we need to carry out in order to decide whether someone understands something or whether someone has a certain competency?
I argue that an important (but by no means sufficient) step towards an operationalization of notions like competency and understanding consists in collecting evidence about typical difficulties students encounter with subject matter. In the case of empty set that would mean that we need to know about students’ difficulties with a particular concept in order to decide whether students understand this concept. If students show certain characteristic difficulties with this concept we can be sure that they do not understand it sufficiently. Unfortunately, it does not tell us that much if students do not show such characteristic difficulties.

Students’ difficulties and competencies are two quite opposite lenses through which students’ learning can be viewed. They are rather complementary to each other. For this reason I argue that we need to complement competency based approaches to curricula by descriptions of students’ difficulties or misconceptions related to relevant topics. Such descriptions are also of paramount importance for effective teaching. Hestenes (1996) suggests the teacher be “equipped with a taxonomy of typical student misconceptions to be addressed as students are induced to articulate, analyse and justify their personal beliefs”. The next section is intended to provide seminal information for such a list related to the concept of empty set.

**Students’ conceptions of the empty set**

In automata theory there are two important concepts of “nothingness”: the empty set and the empty word. The empty word ε is the neutral element of string concatenation (denoted by *), i.e. ε•w=w•ε=w for any word w. Another important construct of the theory is the set of words, called language. For any language L, as for any set, the empty set ø is the neutral element with respect to the union operation, i.e. ø∪L=L∪ø=L.

Note that for any symbol a the operation w•a “adds something” to the word w in that it appends the symbol a to w. Likewise the operation L∪{w} “adds” something to the language L. The analogy, however, is weak: the union operation requires its arguments to be sets. Hence, for adding a word w to a language, w has first to be made an element of a set (w→{w}).

Like any set a language is an unordered collection of words. Words, on the contrary, can be viewed as ordered collections of symbols. For instance, the word foo can be represented as the ordered collection (“f”,”o”,”o”) and

\[
(“f”,”o”,”o”) = (“f”,ε,”o”,”o”) \tag{1}
\]

is a correct statement. Hence, in a way ε is a neutral member of the ordered collection of symbols comprising a word. For sets, however, there is nothing like a neutral member (call it v) such that, for instance,

\[
\{1,2,3\} = \{1,v,2,3\} \tag{2}
\]

holds. In fact such a concept would lead to inconsistencies with the concept of cardinality of a set.
Many students have severe and persistent difficulties to apply the concepts related to $\varepsilon$ and $\emptyset$ and also to tell them apart. Many instructors of automata theory are actually aware of this. However, it is a priori not obvious what causes these difficulties and consequently how to address them effectively in teaching. In order to learn more about this issue I devised the following problem to be used in the very first online formative assessment of an automata theory class for computer science undergraduates (see Kortemeyer (2010) for technical details about the online formative assessments used here).

Problem (N): Which of the following statements about sets are true or false, respectively?

(N1) $\{8,3,5\} = \{3,5,8\}$
(N2) $\{\emptyset,8,3,5\} = \{3,5,8\}$
(N3) $\{\emptyset\} = \emptyset$
(N4) $\{3,\{8,3\},5\} = \{3,5,8\}$
(N5) $\{\{5,1\},\{1,3\}\} = \{\{1,3\},\{1,5\}\}$
(N6) $\{3,\{8,3\},8\} = \{3,8,\{3,8\}\}$

In designing this problem I had been guided by two previous observations: first, that it is unclear to some students that sets are orderless collections (items N1, N5, N6 address this), second, that many students have difficulties to generalise the concept of sets with “atomic” elements to sets of sets (N2-N6 address this). I did not expect students’ problems related to a confusion of $\emptyset$ and $\varepsilon$ at that point, as the empty word was to be introduced later. Also (somewhat naively) I did not expect students’ problems to be related to the empty set alone, as students had to take a course covering set theory as a prerequisite. I felt that this course would have provided sufficient experience related to $\emptyset$. The results of the online formative assessment, however, told me that indeed there are students’ difficulties related to $\emptyset$.

<table>
<thead>
<tr>
<th>Item</th>
<th>N1</th>
<th>N2</th>
<th>N3</th>
<th>N4</th>
<th>N5</th>
<th>N6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial test</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1st try (N = 53, c = 25)</td>
<td>0.96</td>
<td>0.58</td>
<td>0.75</td>
<td>0.83</td>
<td>0.81</td>
<td>0.79</td>
</tr>
<tr>
<td>2nd try (N = 23, c = 9)</td>
<td>0.96</td>
<td>0.52</td>
<td>0.70</td>
<td>0.87</td>
<td>0.83</td>
<td>0.56</td>
</tr>
<tr>
<td>3rd try (N = 14, c = 9)</td>
<td>0.86</td>
<td>0.64</td>
<td>1.0</td>
<td>0.93</td>
<td>0.93</td>
<td>0.79</td>
</tr>
<tr>
<td>Recapitulation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1st try (N = 41, c = 28)</td>
<td>1.0</td>
<td>0.66</td>
<td>1.0</td>
<td>0.93</td>
<td>0.93</td>
<td>0.90</td>
</tr>
<tr>
<td>2nd try (N = 13, c = 8)</td>
<td>1.0</td>
<td>0.62</td>
<td>0.70</td>
<td>1.0</td>
<td>0.77</td>
<td>0.70</td>
</tr>
<tr>
<td>3rd try (N = 5, c = 5)</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Final exam (N = 43)</td>
<td>-a</td>
<td>0.79</td>
<td>0.86</td>
<td>0.93</td>
<td>0.95</td>
<td>-a</td>
</tr>
<tr>
<td>Final exam of subsequent class (N = 62)</td>
<td>-a</td>
<td>0.56</td>
<td>0.67</td>
<td>0.72</td>
<td>0.75</td>
<td>-a</td>
</tr>
</tbody>
</table>

*a Item has not been given in exam.

Table 1. Statistics of the relative number of correct answers for Problem (N). $N$ denotes the number of students having worked on this problem. If there has been more than one possible try $c$ counts the number of students having answered the problem fully correctly in the respective try.
Table 1 lists the results of this online formative assessment within the rubric “Initial test”. Note that items N1-N6 had been encapsulated into one problem such that the online grading engine graded the problem as correct only if all items N1-N6 had been answered correctly. Students whose answers had not been fully correct were granted up to two more trials without being given any hint which of the items N1-N6 had actually been answered correctly or incorrectly. A more detailed analysis of the data given in Table 1 indicates that students were particularly reluctant to change their opinion about item N2. Hence, there might be a misconception related to the underlying mathematics.

Of course my students’ answers on Problem (N) required me to address the observed difficulties with respect to sets in the class session following the online formative assessment. I decided to do this via a peer instruction question cycle (see Mazur (1996) for details on this pedagogy) focusing on items N2 and N3, for peer instruction would allow me to observe what type of arguments my students use in order to justify their answers to this problem. While eavesdropping on my students’ discussion I heard two students justifying their claimed correctness of N2 using an argument along the lines of “∅ isn’t really there, so if you add ∅ as an element it doesn’t change anything”. Note that these students perceived ∅ to have the property of v described by Equation (2). Obviously, they considered ∅ to be the neutral element of a set (in the sense of a neutral member of a set), rather than the neutral element with respect to the union operation.

Having been alarmed about the issue I used some further occasions to address this issue in class. I also gave Problem (N) in another online formative assessment about 8 weeks later and included items N2-N5 in the final exam. The data related to these reassessments (see Table 1 under the rubrics “Recapitulation” and “Final exam”, respectively) suggest that a number of students replaced their preconception related to (2) over the duration of the course. Yet, a considerable number of participants still adhered to (2) in the final exam. A detailed analysis shows that most of these nine students consistently answered N2 incorrectly in all assessments.

In order to rule out that these findings are an artefact of my class and to find out more about this issue I convinced the colleague who taught automata theory in the subsequent semester to include items N2-N5 in his final exam (this colleague was neither using formative assessments nor peer instruction). The corresponding results listed in Table 1 show a remarkable resemblance to the performance of my students on their first try during their very first encounter with these items. These data make it hard to deny that there is some inherent difficulty to this subject matter. This claim is consistent with the findings reported by Fischbein (1994) about other difficulties students typically show in the context of sets.

In order to investigate the issue further, I have started to conduct interviews with students on this issue. These interviews (to be reported elsewhere) strongly confirm that students view the empty set as the neutral member of any set as expressed in Equation (2). For instance, one student clearly uttered: “The empty set is part of every set. […] Therefore within each set one can write the empty set in front of any element.” Another student explicitly tried to justify this by arguing that the empty set is visualised as an
empty Venn diagram. According to this student, the emptiness of the Venn diagram representing $\emptyset$ can be found in the unoccupied space between the elements of any nonempty Venn diagram. Hence, the empty set is an element of any set.

In summary, it appears that many students have a misconception about the properties of the empty set $\emptyset$ in a way symbolically expressed in (2) which is analogous to the correct property (1) of the empty word $\varepsilon$. This might attribute to many students’ difficulties to tell $\emptyset$ and $\varepsilon$ apart.

**Implications for Teaching and Research**

The data presented in the previous section foremost serves the purpose of describing students’ difficulties with the empty set. By following the timeline of events that have led to these results the format of the previous section mimics of storytelling. This format has been chosen on purpose in order to show that uncovering students’ difficulties is completely within the scope of a single instructor in any course. What is not within the scope of a single instructor, however, is to investigate and collect such difficulties on a scale encompassing all mathematical subject matter taught in higher education. This needs to be a collaborative research effort. It is necessary, though, in order for us to gain a better description of what we actually mean by certain content related competencies. In my eyes such an effort is at least as important as efforts to deductively outline competencies.

Taking into account how students’ prior knowledge and beliefs influence learning sheds light on another rather important dimension of competency, namely the competencies of instructors. As emphasised by Schoenfeld (2010), instructors do not only need to have content knowledge and pedagogical knowledge, but also pedagogical content knowledge. That is, they need to know how student learning can be fostered based on knowledge about their students’ difficulties with subject matter and what makes this subject matter difficult. Given how little is known today about why certain subject matter is difficult to learn, instructors need to be able to elicit students’ preconceptions and difficulties. Appropriate tools for acquiring pedagogical content knowledge “on the fly” are conveniently accessible today. Formative assessments are an example for such tools. Formative assessments also had been the starting point for the deeper investigation of students’ conceptions of empty set and empty word as described in the previous section. On a slightly more abstract level it relates to what Aarons (1974) already described decades ago: “I am deeply convinced that a statistically significant improvement would occur if more of us learned to listen to our students. […] By listening to what they say in answer to carefully phrased, leading questions, we can begin to understand what does and does not happen in their minds, anticipate the hurdles they encounter, and provide the kind of help needed to master a concept or line of reasoning without simply ‘telling them the answer.’”
References


