

# Computer Algebra System Supported Environment for Teaching and Learning Calculus I

Tsvetanka Kovacheva

*Kaunas University of Technology, Kaunas, Lithuania*

## Abstract

The paper presents a computer algebra system (CAS)-supported approach for teaching and learning Mathematics applying Matlab as a symbolic, numeric and graphic tool. It is used in education of some specialties of the bachelor's degree studies at the Technical University (TU) – Varna, Bulgaria. The students' activities in the Calculus I laboratory exercises to evaluate limits and derivatives are discussed. The evaluation of the students in laboratory exercises is considered.

## Introduction<sup>1</sup>

The development of computer technologies allowed the possibility of providing interactive software for CAS - Matlab, Mathematica and others. They have been widely adopted in the academic community to make education more authentic and tangible. More than 3 500 universities worldwide apply CAS as training and research tools. The great power and potential of CAS are addressed to mathematics lecturers' and students' activities. However the implemented symbols, notions, functions and terminology in Mathematics cannot help to automatically transfer mathematical knowledge to students and to enhance their feeling for Mathematics. Methodological principles and lecturer's capabilities to put these principles into practice in a CAS-supported environment can give CAS vitality.

CAS is also used in the Mathematics study process during the Bachelor's Degree in the TU. Almost all first and second-year students at the university have laboratory exercises (labs) as part of their high mathematics (HM) education. The students learn three parts of mathematics. These are realized with the assistance of CAS - Matlab and Mathematica (for the specialism of Electronics). The university provides a thorough base for conducting education. It holds specially-equipped laboratories. The Internet link allows access to Internet-based resources for the Mathematics tuition. The aims of labs are to enable students to solve given problems during lectures and seminars using a ready-made program product on the computer. One of the most recent versions of CAS is in use. It allows the application of a dialogue regime of operation, a convenient interface and mathematical calculation media. There is a help menu for the applied calculation commands, the graphic visualization of the calculations and for programming.

Since the academic year 2004/2005 tuition for Turkish students has been launched at TU. The Bachelor's degree tuition has been carried out in the following specialties: Computer Science and Engineering (CSE), Electronics (E), Industrial Management

---

<sup>1</sup> See Company Mathworks (Matlab) (2008), Kovacheva (2004, 2007).

(IM), Ship Construction and Marine Equipment (SCME) and Social Management (SM). Foreign students are taught to the Bulgarian curricula for the respective specialism, designed in the Mathematics Department. Mathematics classes are scheduled for 3 semesters for CSE, SCME and E, for 2 semesters for IM. Students in SM do not study Mathematics. Students in their first and second year attend separately designed lectures, seminars and lab classes in English. Labs are carried out utilizing CAS: Matlab for CSE, SCME and E, and Matlab for E.

### Some Topics in Calculus I labs using Matlab<sup>1</sup>

The Symbolic Math Toolbox of Matlab is used to realize the Calculus I labs. As examples the basic students' activities in two labs for evaluating limits and derivatives of a one variable are given. At the beginning of each lab a short theoretical review is presented.

#### Limit of a function of one variable Methodology

In this lab students learn how to use Matlab for finding limits of functions of one variable. The system performs the calculation of most limits very quickly by the command **limit**. Function graphics are used to verify the correctness of the defined limits. The activities of the students are the following:

##### *Activity 1. Evaluating limits exploring $\varepsilon - \delta$ definition*

The students can understand the evaluating process if they solve the problems using the  $\varepsilon - \delta$  definition. For example they determine a value of  $\delta$  so that if  $|x - 1| < \delta$

then  $\left| \frac{x}{x+1} - \frac{1}{2} \right| < \varepsilon$ ,  $\varepsilon = 0.001$ . The students solve the inequality  $\left| \frac{x}{x+1} - \frac{1}{2} \right| - 0.001 < 0$  near  $x = 1$ .

##### *Activity 2. Evaluate limits using the table of function values near the limiting point and the function graph*

The students consider two fundamental limits:  $\lim_{x \rightarrow 1} \frac{\sin(x)}{x} = 1$  and  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$ . They

plot graphs of the functions  $y = \frac{\sin x}{x}$  and  $y = \left(1 + \frac{1}{x}\right)^x$  (Figure 1, Figure 2) and find the tables of functions values near the points 0 and  $e$ , respectively (Table 1). NaN ("Not a Number") value opposite the point 0 denotes that the function value is undefined. It

---

<sup>1</sup> See Calculus with Matlab (2008), Educational Mathematical Site Exponenta (2008), Nikolaev & Chakarov (1999), Tonchev (2005).

appears from the Table 1 that if  $x$  is near 0 then  $y = \frac{\sin x}{x}$  is close to 1. The same result is obtained for the graph of the function.

x	$y = \frac{\sin x}{x}$	x	$y = \left(1 + \frac{1}{x}\right)^x$
-0.03	0.999850	50 000	2.718254646126732
-0.02	0.999933	51 000	2.718255179101404
-0.01	0.999983	52 000	2.718255691602668
0	NaN	53 000	2.718256184737094
0.01	0.999983	54 000	2.718256659595320
0.02	0.999933	55 000	2.718257117232703
0.03	0.999850	56 000	2.718257558471690

Table 1

The students will see what happened near the origin of the function  $y = \left(1 + \frac{1}{x}\right)^x$  (Table

1, Figure 2). The limit exists and its value is one of the most important numbers in mathematics. To 15 digits the correct value is 2.718281828459045.

*Activity 3. Evaluate limits using **limit** command*

The students evaluate the limits of different groups of functions: rational, irrational, trigonometric, etc. At first they declare  $x$  as a symbolic variable and use the **limit** command to evaluate the limits. For example:  $\lim_{x \rightarrow \pi/2} \sin(x)$

```
>> syms x
```

```
>> limit(sin(x),pi/2)
```

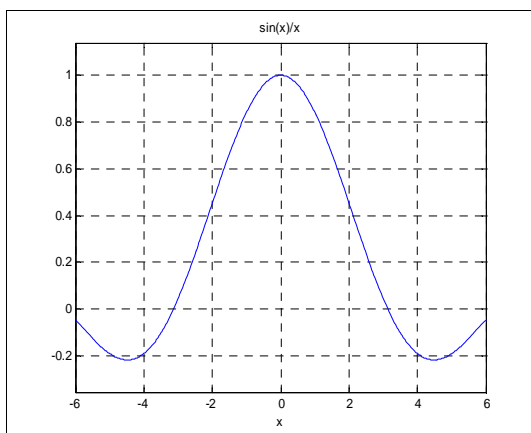


Figure 1

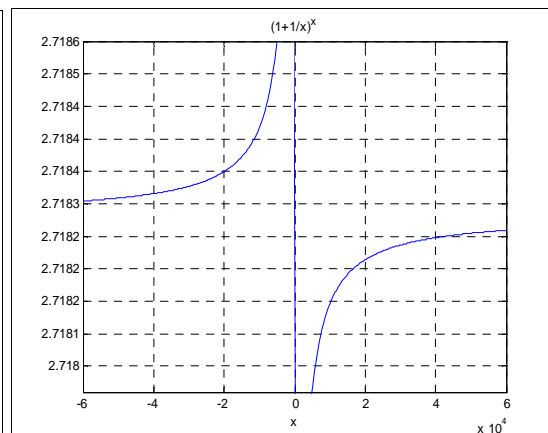


Figure 2

*Activity 4. Evaluate some non existing limits*

In the case when the limit at a specific point has not been defined, the Matlab cannot find it and the value is undefined - NaN. For the functions  $f(x) = 1/x^2$  and  $f(x) = 1/x^3$  using the commands

```
>>limit(1/x^2,0)
```

```
>>limit(1/x^3,0)
```

the students obtain the answer infinity in the first case and NaN in the second. They plot the graph of the functions under consideration (Figure 3, Figure 4).

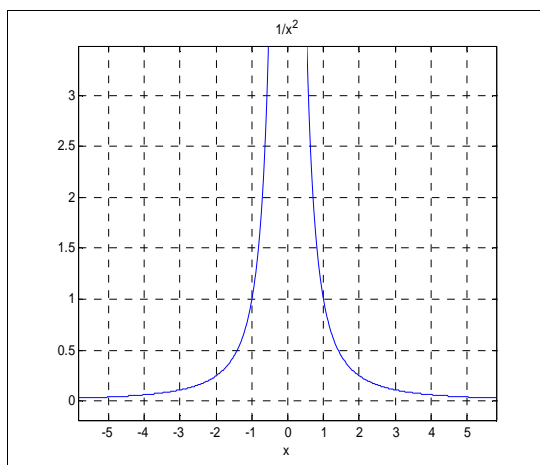


Figure 3

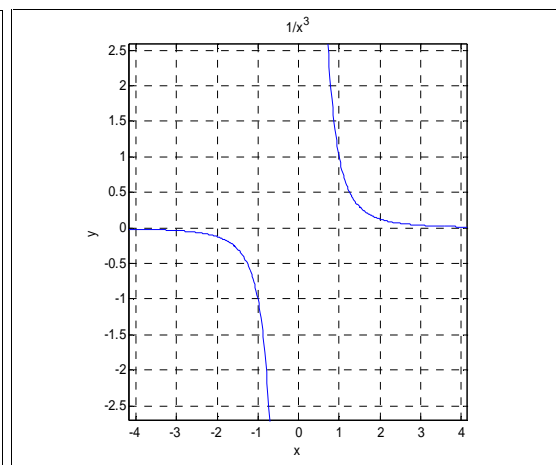


Figure 4

*Activity 5. Evaluate one-sided limits of the function*

The students investigate what happens to the function  $f(x) = 1/x^3$  when  $x$  approaches 0 on the left and on the right. They specify the direction of approach. The commands are:

```
>>limit((1/x^3,x,0,'right')
```

```
>>limit((1/x^3,x,0,'left')
```

The left limit is minus infinity and the right is infinity (Figure 4).

It is well to show of the students similar problems as in Figure 5. The first three functions have limit (-5) as  $x$  approaches 1 and the fourth function has different left and right limits at 1, and so the limit does not exist.

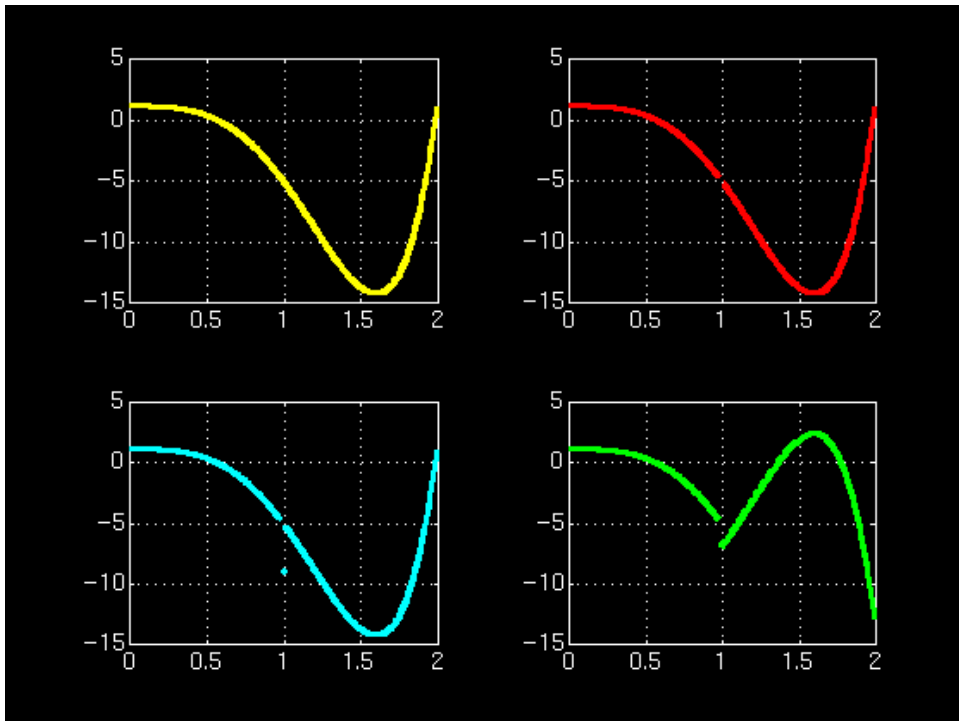


Figure 5

### Symbolic differentiation

The other important notion for the students is derivative of a function. The main idea is they to learn the definition of derivative using **limit** command and the Matlab **diff** command. **The basic activities for the students are:**

*Activity 1. Define analytically the derivative of simple function and its value at a fixed point*

*Activity 2. Calculate the values of the derivative numerically*

Consider the function

$$F(x) = \frac{f(x) - f(a)}{x - a} \quad (1)$$

Using Matlab command **inline** function  $g(x)$  is created for the function (1) and evaluated for series of values close to a fixed value  $a$ .

*Activity 3. Calculate the derivative using the definition*

The students evaluate the derivative using the **limit** function

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (2)$$

*Activity 4. Calculate the derivative using the Matlab **diff** command*

Using the command **diff** the students evaluate different elementary and complex functions.

*Activity 5. Compare the obtained values by the different approaches*

*Activity 6. Repetition of activities 2 to 3 for different value of  $a$ , resp.  $x$  and representation the results graphically.*

*Activity 7 Geometrical interpretation of the derivative*

The value of (2) is the slope of the tangent line to the graph of  $f$  at the given point of  $(x_0, f'(x_0))$ . The equation of the tangent is  $y = f'(x_0).(x - x_0) + y_0$ .

The results of the above students' activities for the function  $f(x) = x^3 + 2.x^2 + 1$  and  $x=1$  are the following. They students construct an inline function for the  $F(x)$

```
>g=inline('(x^3+2.x^2+1)-4)/(x-1)')
```

The next step they evaluate the function  $g$  for several values of  $x$  and find  $g(1.1)$

```
>g(1.1)
```

```
ans = 5.71
```

Then using the **limit** or **diff** the students determine exact value of (1) or (2), when  $x=1$  with the commands

```
> limit((x^3+2.x^2+1)-4)/(x-1),x,1)
```

```
> d=diff(x^3+2.x^2+1,x);subs(d,1)
```

and obtain that derivative of a function is  $f'(x) = 3.x^2 + 4.x$  and  $f'(1) = 7$ .

After that they find the values of  $g$  for given values of  $x$  close to  $1$ . Analogy are imputed the inline functions for  $F(x)$  when  $x=2$  and  $3$ , respectively and are evaluated their values for given values of  $x$  close to  $2$  and  $3$  (Table 2, Figure 6).

a=1		a=2		a=3		a=4		a=5		a=6	
x	F(x)	x	F(x)	x	F(x)	x	F(x)	x	F(x)	x	F(x)
1.01	7.05	2.01	20.08	3.01	39.11	4.01	64.14	5.01	95.17	6.01	132.20
1.10	7.51	2.10	20.81	3.10	40.11	4.10	65.41	5.10	96.71	6.10	134.01
1.30	8.59	2.30	22.49	3.30	42.39	4.30	68.29	5.30	100.19	6.30	138.09
1.50	9.75	2.50	24.25	3.50	44.75	4.50	71.25	5.50	103.75	6.50	142.25
2.00	13.00	3.00	29.00	4.00	51.00	5.00	79.00	6.00	113.00	7.00	153.00
3.00	21.00	4.00	40.00	6.00	81.00	6.00	96.00	7.00	133.00	8.00	176.00
4.00	31.00	5.00	53.00	6.00	81.00	7.00	115.00	8.00	155.00	9.00	201.00

Table 2

x	f'(x)
1	7
2	20
3	39
4	64
5	95
6	132
7	175

Table 3

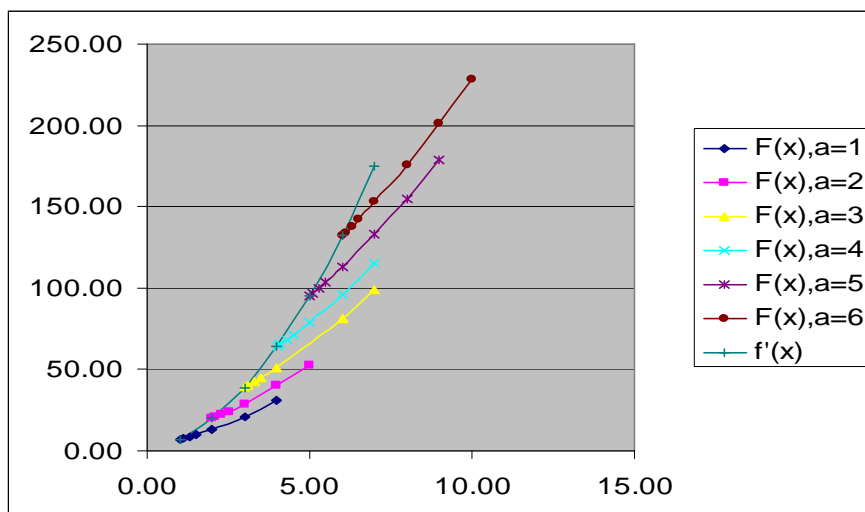


Figure 6

The students find the equation of the tangent to (2) at the point  $(1, f(1))$  -  $y = 7 \cdot (x - 1) + 4$  and plot the results.

### Evaluation the Students' Activities in the Labs

The lecturer gives to students similar problems to solve and these are graded at the end of the lab. During the lab he ensures that the study process is carried out in the right manner. Checks and instructions are carried out for the whole group to economize time. Discourse and discussions facilitate the build-up of long-term and in-depth knowledge.

At the end of lab each student has an individual check with the lecturer on the validity of his solutions to the given problems; students are then aided in the drawing of conclusions and the evaluation of their work. The final assessment of each student is complex and takes into account their progress during labs, marks from the progress checks and/or written assignments.

To achieve better results from the execution of the laboratory classes and to make the system and exercises more appealing to the students, the lecturer can use different materials, such as a CD with the Matlab version, printed matter of the system operation, problems to be set aside for homework, assigned term papers.

## Conclusion

Combining traditional modes and CAS-supported approaches to undergraduate mathematics during labs enhances the educational effectiveness.

Matlab holds great potential in assisting the study process of the chapters included in the mathematics tuition. Every lecturer aims at being most collaborative with their students and in order to achieve efficiency he/she has to creatively make use of the system's capabilities. With the lecturer's consideration, depending on experience, various exercises to suit the purposes of the labs can be performed. Such exercises could only enhance the in-depth comprehension of the material taught and likewise motivate students' interaction.

The application of Matlab in labs allows for a greater amount of solved problems among students, helps them determine certain properties unaided and thence comprehend the content of notions more thoroughly, their properties, theorems and the relations between them. Students learn to independently solve problems with the help of a CAS and to select methods for their solving. Mastering Matlab during the first years at University is beneficial to students not only in solving mathematical problems but also any complex applied technical problems they will encounter.

## References

- Calculus with Matlab (2008) <http://www.cs.bgu.ac.il/~berend/teaching/Numerical-Analysis/MATLAB/Calculus%20with%20MATLAB.htm> (1 April 2008).
- Company Mathworks (Matlab) (2008) <http://www.mathworks.com> (1 April 2008).
- Educational Mathematical Site Exponenta (2008) <http://www.exponenta.ru/> (1 April 2008).
- Kovacheva, T. (2007) "English-Taught Mathematics at the Technical University – Varna". In: *XXXVI spring conference of UBM*, Varna, 2-6 .04.2007.
- Kovacheva, T. (2004) Mathematical packages for teaching and research in internet – application and information support. *Information Theories & Applications*, vol.11, 66-69.
- Nikolaev, R., Chakarov, E. (1999) *Mathematical analysis*, Varna.
- Tonchev, J. (2005) *Matlab, part I and II*. Technika, Sofia (in Bulgarian).