Lagrange Multipliers as Quantitative Indicators in Economics

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Abstract

The quantitative role of Lagrange multipliers is under consideration. Applications in economics are examined. Illustrative examples are presented.

Introduction

The Lagrange multipliers method is readily used for solving constrained extrema problems. Let us concentrate on the rationale for this method. Recall that for a function $f$ of $n$ variables the necessary condition for local extrema is that at the point of extrema all partial derivatives (supposing they exist) must be zero. There are therefore $n$ equations in $n$ unknowns (the $x$'s), that may be solved to find the potential extrema point (called critical point). When the $x$'s are constrained, there is (at least one) additional equation (constraint) but no additional variables, so that the set of equations is overdetermined. Hence the method introduces an additional variable (the Lagrange multiplier), that enables to solve the problem. More specifically (we may restrict to finding a maxima), suppose we wish to find values $x_1, \ldots, x_n$ maximizing

$$ y = f(x_1, \ldots, x_n) $$

subject to a constraint that permits only some values of the $x$'s. That constraint is expressed in the form

$$ g(x_1, \ldots, x_n) = 0. $$

The Lagrange multipliers method is based on setting up the new function (the Lagrange function)

$$ L(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) + \lambda g(x_1, \ldots, x_n), $$

where $\lambda$ is an additional variable called the Lagrange multiplier. From (1) the conditions for a critical point are

$$ L'_{x_1} = f'_{x_1} + \lambda g'_{x_1} = 0 $$

$$ \ldots $$

$$ L'_{x_n} = f'_{x_n} + \lambda g'_{x_n} = 0 $$

$$ L'_{\lambda} = g(x_1, \ldots, x_n) = 0, $$

where the symbols $L'$,$g'$ are to denote partial derivatives with respect to the variables listed in the indices. Of course, equations (2) are only necessary conditions for a local maximum. To confirm that the calculated result is indeed a local maximum second order conditions must be verified. Practically, in all current economic problems there is on economic grounds only a single local maximum.

In a standard course of engineering mathematics the Lagrange multiplier is usually presented as a clever mathematical tool ("trick") to reach the wanted solution. There is no large
spectrum of sensible examples (mostly a limited number of simple “well-tried” school examples) to show convincingly the power of the method. The economic meaning of the Lagrange multiplier provides a strong stimulus to strengthen its importance. This will be central to our next considerations.

Economic milieu

To grasp the issue we will notice two useful meanings \(i^0, 2^0\) of the Lagrange multipliers.

1. Rearrange the first \(n\) equations in (2) as

\[
\frac{f'}{x_i} = \ldots = \frac{f'}{x_n} = \lambda. \tag{3}
\]

Equations (3) say that at maximum point the ratio of \(f'\) to \(g'\) is the same for every \(x_i\) and moreover it equals \(\lambda\). The numerators \(f'\) give the **marginal contribution** (or **benefit**) of each \(x_i\) to the function \(f\) to be maximized, in other words they give the approximate change in \(f\) due to a one unit change in \(x_i\). Similarly, the denominators have a marginal cost interpretation, namely, \(-g'\) gives the **marginal cost** of using \(x_i\) (or marginal “taking” from \(g\)), in other words the approximate change in \(g\) due to a unit change in \(x_i\). In the light of this we may summarize, that \(\lambda\) is the **common benefit-cost ratio** for all the \(x's\), i.e.

\[
\lambda = \frac{\text{marginal contribution of } x_i}{\text{marginal cost of } y_i} = \frac{f'x_i}{-g'x_i} \tag{4}
\]

Example Let \(Q = Q(l,k)\) be a production function, where \(l\) is a labour and \(k\) capital. The cost to the firm of using as input \(l\) units of labour and \(k\) units of capital is

\[P_l l + P_k k,\]

where \(P_l\) and \(P_k\) are the per unit costs of labour and capital respectively. If the firm has a fixed amount, \(M\), to spend on these inputs then the cost constraint is

\[P_l l + P_k k = M.\]

In order to maximize the function \(Q(l, k)\) subject to this constraint we set up the Lagrange function (rewriting constraint condition to \(M - P_l l - P_k k = 0\))

\[L(l, k, \lambda) = Q(l, k) + \lambda(M - P_l l - P_k k).\]

Due to (2) it holds

\[
\begin{align*}
L'_{l} &= Q'_{l} - \lambda P_{l} = 0 \\
L'_{k} &= Q'_{k} - \lambda P_{k} = 0 \\
L'_{\lambda} &= M - P_{l} l - P_{k} k = 0,
\end{align*}
\]

but \(Q'_{l} = M_{l}\) is the marginal product of labour and \(Q'_{k} = M_{k}\) is the marginal product of capital. Then first two equation can be rearranged (according to (3)) to give
which states that at the maximum point the ratio of marginal product to price is the same for both inputs and it equals $\lambda$.

Example A farmer has a given length of fence $F$ and wishes to enclose the largest possible rectangular area. The question is about the shape of this area. To solve it, let $x, y$ be lengths of sides of the rectangle. The problem is to find $x$ and $y$ maximizing the area $S(x, y) = xy$ of the field, subject to the condition (constraint) that the perimeter is fixed at $F = 2x + 2y$. This is obviously a problem in constraint maximization. We put $f(x, y) = S(x, y)$, $g(x, y) = F - 2x - 2y = 0$ and set up the Lagrange function (1)

$$L(x, y, \lambda) = xy + \lambda(F - 2x - 2y).$$

Conditions (2) are

$$L_x' = y - 2\lambda = 0, L_y' = x - 2\lambda = 0, L_{\lambda}' = F - 2x - 2y = 0.$$

These three equations must be solved. The first two equations give $x = y = 2\lambda$, i.e. $x$ must be equal to $y$ and due to (5) they should be chosen so that the ratio of marginal benefits to marginal cost is the same for both variables. The marginal contribution to the area of one more unit of $x$ is due to (4) given by $S'_x = y$ which means that the area is increased by $y$. The marginal cost of using $x$ is $-g'_x = 2$. It means value 2 from $g$; but since $g(x, y) = F - 2x - 2y$, the value 2 is taken from the available perimeter $F$. As mentioned above, the conditions (4) state that this ratio must be equal for each of the variables. Completing the solution (substituting $x = y = 2\lambda$ in $F - 2x - 2y = 0$) we get $\lambda = \frac{F}{8}, x = y = \frac{F}{4}$. Now let us discuss the interpretation of $\lambda$. If the farmer wants to know, how much more field could be enclosed by adding an extra unit of the length of fence, the Lagrange multiplier provides the answer $\frac{F}{8}$ (approximately), i.e. the present perimeter should be divided by 8. For instance, let 400 be a current perimeter of the fence. With a view to our solution, the optimal field will be a square with sides of lengths $\frac{F}{4} = 100$ and the enclosed area will be 10 000 square units. Now if perimeter were enlarged by one unit, the value $\lambda = \frac{F}{8} = \frac{400}{8} = 50$ estimates the increase of the total area. Calculating the “exact” increase of the total area, we get: the perimeter is now 401, each side of the square will be $\frac{401}{4}$, the total area of the field is $(\frac{401}{4})^2 = 10050.06$ square units. Hence, the prediction of 50 square units given by the Lagrange multiplier proves to be sufficiently close.

Example Let an individual’s health (measured on a scale of 0 to 10) be represented by the function $f$,

$$f(x, y) = -x^2 + 2x - y^2 + 4y + 5,$$
where \( x \) and \( y \) are daily dosages of two drugs. It may be verified, that this function attains its (local) maximum for \( x = 1, y = 2 \) with the corresponding value of \( f(1,2) = 10 \). So, at that point is the best health status 10 possible. Now we want to maximize \( f \) under the constraint that this individual could tolerate only one dose per day, i.e. \( x + y = 1 \). We put \( f(x, y) = -x^2 + 2x - y^2 + 4y + 5 \), \( g(x, y) = 1 - x - y = 0 \) and set up the Lagrange function

\[
L(x, y, \lambda) = -x^2 + 2x - y^2 + 4y + 5 + \lambda(1 - x - y).
\]

Conditions (2) are

\[
L' = -2x + 2 - \lambda = 0, L'_y = -2y + 4 - \lambda = 0, L'_\lambda = 1 - x - y = 0.
\]

Applying Lagrange multipliers method we get the solution \( x = 0, y = 1, \lambda = 2 \). with the corresponding value of \( f(0,1) = 8 \). The value 2 may be interpreted as the remainder to the maximum value of health status 10. Now we reduce the restriction altering the constraint equation to \( x + y = 2 \). We expect \( f \) to increase. Finding the new solution as before we have \( x = 0.5; y = 1.5; \lambda = 1 \) with \( f(0.5; 1.5) = 9.5 \). So, there is still some remainder (approximately \( \lambda \approx 1 \), precisely 0.5) to the optimal health status. Further reducing constraint to \( x + y = 3 \) leads to the solution \( x = 1, y = 2, \lambda = 0 \) which is the maximum of \( f \) (without constraint). For higher sums of \( x + y \) (overdose) we expect negative values of \( \lambda \).

2\(^{0}\) Rewrite constraint condition \( g(x, y) = 0 \) as \( c(x, y) = k, g(x, y) = k - c(x, y) = 0 \), where \( k \) is a parameter. Then the Lagrange function is of the form

\[
L(x, y) = f(x, y) + \lambda (k - c(x, y)).
\]

For the partial derivative of \( L \) with respect to \( k \) we get \( L'_k = \lambda \). From the interpretation of a partial derivative we conclude, that the value \( \lambda \) states the approximate change in \( L \) (and also \( f \)) due to a unit change of \( k \). Hence the value \( \lambda \) of the multiplier shows the approximate change that occurs in \( f \) in response to the change in \( k \) by one in the condition \( c(x, y) = k \), i.e. \( c(x, y) = k + 1 \). Since usually \( c(x, y) = k \) means economic restrictions imposed (budget, cost, production limitation), the value of multiplier indicates so called the opportunity cost (of this constraint). If we could reduce the restriction, ie. to increase \( k \) by 1, then the extra cost is \( \lambda \). If we are able to realize an extra unit of output under the cost less than \( \lambda \), then it represents the benefit due to the increase of the value at the point of maxima. Clearly to the economic decision maker such information on opportunity costs is of considerable importance.

**Example** The profit of some firm is given by \( PR(x, y) = -100 + 80x - 0.1x^2 + 100y - 0.2y^2 \), where \( x, y \) represent the levels of output of two products produced by the firm. Let us further assume that the firm knows its maximum combined feasible production to be 500. It represents the constraint \( x + y = 500 \). Putting \( g(x, y) = 500 - x - y = 0 \) we set up the Lagrange function

\[
L(x, y, \lambda) = -100 + 80x - 0.1x^2 + 100y - 0.2y^2 + \lambda(500 - x - y).
\]

Applying the Lagrange multipliers method we get the solution \( x = 300, y = 200, \lambda = 20 \) with the corresponding value of the profit \( PR(300,200) = 26900 \). Now we reduce the restriction altering the constraint equation to \( x + y = 501 \). Finding the new solution as before we have
\[
\begin{align*}
x &= \frac{902}{3} = 300,666, \\
y &= \frac{601}{3} = 200,333, \\
PR &= PR\left(\frac{902}{3}, \frac{601}{3}\right) = 26919,933.
\end{align*}
\]
We see that the increase in profit brought about by increasing the constraint restriction by 1 unit has been 19,933 - approximately the same as the value \(\lambda\) that we derived in the original formulation. It indicates that the additional increase of labour and capital in order to increase the production has the opportunity cost of approximately 20.

Example The utility function is given by \(U = U(x,y) = 4x^{0.5}y^{0.25}\), where \(x\) or \(y\) is the number of units of a good \(X\) or \(Y\) respectively. Suppose the price of \(X\) is 2.5 USD per unit, the price of \(Y\) is 4 USD per unit. To calculate the optimal combination for an income of 50 USD we employ Lagrange multipliers method. The constraint is given by \(2.5x + 4y = 50\). We put \(g(x,y) = 50 - 2.5x - 4y\) and form Lagrange function

\[
L(x, y) = 4x^{0.5}y^{0.25} + \lambda(50 - 2.5x - 4y).
\]

Applying this method we get \(x = \frac{40}{3}, y = \frac{25}{6}, \lambda = 0.313\) with the corresponding value of utility \(U\left(\frac{40}{3}, \frac{25}{6}\right) = 20,867\). Now we moderate the constraint to \(2.5x + 4y = 51\). Applying the method again we obtain the solution \(x = \frac{68}{5}, y = 4.25\) with the corresponding value of utility \(U\left(\frac{68}{5}; 4.25\right) = 21,180\). We see that the increase in utility equals the value of \(\lambda\).

Conclusion

In the instruction of engineering mathematics the Lagrange multipliers method is mostly applied in cases when the constraint condition \(g(x,y) = 0\) cannot be expressed explicitly as the function \(y = f(x)\) or \(x = h(y)\). When solving constrained extrema problems in economics the bulk of the constraint conditions may be expressed explicitly, so the reason to use the Lagrange multipliers method would seem to be too sophisticated regardless of its theoretical aspects. With a view to the crucial importance of the economic interpretations of Lagrange multipliers is the use of the method primarily preferred. Concrete applications of the presented interpretation principle may be developed in many economic processes. A deeper study on the role of the Langrange multipliers in optimization tasks may be found in Rockafellar (1993).

References


