TREE – Teaching and Research in Engineering in Europe
Special Interest Group B7 "Mathematics education for engineers in the changing world "
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# Mathematics education for engineers in the changing world

## INTRODUCTION

Engineering has always been underpinned by mathematics as a language both for the expression of ideas and a means of communicating results. Furthermore, mathematical thinking gives engineers a means of formulating, analyzing and solving a wide range of practical problems. The current rapid pace of technological change has increased the importance of mathematics to engineering.

Alongside this rapid technological change, there have been significant social changes particularly in terms of the availability of higher education, the impact of the internet on student approaches to learning, the attractiveness to teenagers of certain academic disciplines and the implementation of the Bologna declaration. As a consequence of these things, what mathematics to teach to both undergraduate and graduate engineering students and how to teach it are pressing problems of considerable importance. Mathematics education should enable engineering students to communicate their ideas in an unambiguous and understandable way and should equip them with the analytical skills they will need as practicing engineers. But mathematics is more than just a collection of tools that can be used to solve certain well-defined problems. Mathematical thinking and modeling give engineers the ability to approach new problems with confidence.

Main problems relating to the mathematical education of engineers are:

- the tension between ensuring basic skills are mastered and developing conceptual thinking and modeling skills;
- appropriate use of new software (such as computer algebra systems) so that mathematical education is improved;
- developing assessment methods that focus on higher level abilities not just routine application of standard methods.

There are several important issues that are currently being faced by those involved in the mathematical education of engineers. These issues are often closely interlinked. In the list below, they have been grouped into a small number of over-arching headings. However, due to the interlinking previously referred to, some items could have reasonably have been placed under other headings.

**1 Integration and Co-operation**

In practice, the interface between mathematics and engineering is not clearly defined. However, in the students' minds there is often a very clear division with mathematics being thought of as a closed, complete body of knowledge which is essentially abstract but which may be used as a tool (or as a series of tools) to solve certain kinds of engineering problems. A good mathematical education will challenge such thinking in a number of ways including:

- Engineers should view mathematics as a way of thinking and communicating rather than as a set of tools.

- Students need to develop skills in using mathematics to solve problems and they also need to see mathematics as an integral part of engineering applications and not as residing in a separate compartment.

- Mathematical modelling is a fundamentally important engineering activity which is neither purely abstract nor necessarily deriving from an existing complete body of knowledge.

- Engineers may sometimes need to develop new mathematics in order to solve certain problems (for example, fuzzy logic and finite element methods).

To address these issues requires good co-operation between mathematics staff and their colleagues teaching engineering applications.

**2 Attraction**

There are many time pressures on the engineering curriculum – not only from engineering applications but also from other subjects such as business, finance and foreign languages. In the face of this competition it is important that:

- Mathematics advertises itself – that mathematicians promote what they do and how it makes engineers into better engineers.

- The mathematics required in the workplace is fully considered by involving practising engineers in the development of new curricula.

- Particularly at Masters level, where students have the opportunity to take option modules that allow them to follow their specific interests, it is vital that attractive units in engineering mathematics are available.

## 3 Attitudes to Teaching and Learning

The expansion of higher education across Europe has brought with it a change in the general attitude of the student body. Many students attend university with a primary objective of 'getting a degree' rather than to learn about the subject they are studying. This brings with it a series of challenges which may perhaps be addressed by adopting new approaches to teaching.

- There is a need to engage students so that they actively participate in learning. At a basic level, in many institutions, there is a need to promote greater attendance at classes. At a higher level, in most institutions there is a need to move students from passive recipients of knowledge (people who have education 'done to them') to active gatherers of understanding (people who seek out learning).

- Assessment has a key role to play in determining student approaches to learning. Much student activity is driven by assessment. So, it is essential that we assess those things that we want students to learn, such as mathematical thinking. This is often considerably more difficult than simply assessing mastery of a particular mathematical technique.

- The issue of how the mathematics curriculum needs to change in the light of technological advances is still being debated. For example, widely available software will now compute integrals in an instant in response to a simple one line command but integration should not disappear from engineering mathematics curricula. The focus of what is covered in a study of integration may however be considerably different from what it was a few years ago.

- Implementation of the Bologna Declaration has produced several unforeseen difficulties for mathematics. In particular, the number of contact hours available for mathematics teaching in 3 year bachelors programmes has been greatly reduced compared to the 5 year programmes they have replaced. Many Masters programmes do not contain any further mathematics and, as a consequence, all engineers (both bachelors and masters) receive considerably less mathematics teaching than they did before the implementation of Bologna.

## 4 General Issues

The issues discussed below do not only apply to the mathematical education of engineers but instead are relevant across much of higher education. They are however, highly relevant to engineering mathematics and so need to be considered in any discussion of current issues in this area.

- The importance attached to good teaching in comparison to good research – many universities recruit staff primarily because of their research record not because they are good teachers. Pressures to produce research publications limits the time that many staff are willing to spend to develop their teaching.

- The increasing number of students studying in higher education has increased the inhomogeneity of undergraduate cohorts. The gap in background knowledge between the best prepared students and the least well-prepared students is often so great that it is very difficult to develop an appropriate teaching strategy.

- The lecture is still the primary teaching instrument in many universities (particularly for engineering mathematics) and this may not be particularly suitable for freshmen who are used to shorter, less intensely concentrated learning experiences.

- Many students undertake several hours of part-time paid employment in order to help finance themselves and this limits the number of hours they can devote to their studies.

In what follows we concentrate on the problem how to change the curriculum in mathematics after the changes caused by the Bologna declaration.

## BOLOGNA PROCESS AND CORE CURRICULUM IN MATHEMATICS

Splitting long engineering programmes into two programmes lead in many European countries to decrease number of contact hours for mathematics teaching. The implications of Bologna for the curriculum must be considered, as well as the likely effects this would have on the mathematical education of engineers. The Agreement calls for Bachelor programmes of 180 European Credits (ECTS), taken over about 3 years or 6-7 semesters, followed by 120 ECTS up to masters level (taking about 2 years or 4 semesters). The decision to opt for this was political and dictated by the aim of transference in academic study between European institutions.

In 2002 the SEFI Mathematics Working Group produced "MATHEMATICS FOR THE EUROPEAN ENGINEER, A Curriculum for the Twenty-first Century" [1]. This document divided the necessary knowledge of a future engineer into different levels. The Core Zero contains the basic knowledge that should be gained during the secondary education; Core Level One brings the content of the first years of engineering study. Level Two is devoted to the list of advanced mathematical topics that different branches of engineering students should get. [1] was prepared during the time when most of the European Higher Education Institutions had so called "long" programmes leading directly to an engineering degree. The changes due to Bologna process, mainly splitting the long programmes into two consecutive self-contained programmes, have brought significant changes also into the syllabuses and some modifications of [1] are needed. We bring a suggestion of such modifications. Note, that we bring only a modifications to [1], the material mentioned above.

## Core Zero

Core Zero, which will be specified later, comprises essential material that no engineering student can afford to be deficient in. Since now the knowledge and skills in mathematics of a student entering bachelor programme are not easy to predict (and it varies a lot) this must be achieved either before entering a bachelor programme or through some support during the first year of bachelor degree programme. The part that contains basics of Calculus of one variable is not covered by secondary education in all European countries, in such cases the first year mathematics education should be enlarged by this content.

### 1. Algebra

Arithmetic of real numbers

The student should be able to
- carry out operations with integers, including division of integers
- understand a concept of a prime, express an integer as a product of prime factors

- calculate the greatest common divisor and the least common multiple from the factorization into primes
- understand the rules governing the existence of powers of a number
- carry out operations with fractions
- work with negative powers in the set of rational numbers
- express a rational number in decimal form and carry out operations with rational numbers in decimal form
- round numerical values of a rational number to a specified number of decimal places or significant figures
- understand the scientific notation form of a number
- work with logarithms
- understand how to estimate errors in measurements and how to combine them

Algebraic expressions and formulae

The student should be able to
- add, subtract and multiply algebraic expressions and simplify the result
- evaluate algebraic expressions
- changes the subject of a formula
- distinguish between an identity and an equation
- obtain the solution of a linear equation
- solve two simultaneous equations both algebraically and geometrically
- understand the terms direct proportion and inverse proportion
- solve simple problems involving proportion
- factorize a quadratic expression
- carry out the operations add, subtract, multiply and divide on algebraic fractions
- interpret simple inequalities in terms of intervals on the real line
- solve simple inequalities, both geometrically and algebraically
- understand the absolute value of a number as a distance on real line of the origin
- interpret inequalities which involve the absolute value

Linear laws

The student should be able to
- understand the Cartesian co-ordinate system
- plot points using Cartesian co-ordinates
- understand the term “gradient” and “intercept” of a straight line
- obtain and use the equation of a line with know gradient through a given point
- obtain and use the equation of a line through two given points
- use the general equation $ax + by + c = 0$
- determine algebraically whether two points lie on the same side of a straight line
- recognise when two lines are parallel
- recognise when two lines are perpendicular
- interpret simultaneous linear inequalities in terms of regions in the plane

Quadratics, cubics, polynomials

The student should be able to
- recognise the graphs of $y = x^2$ and $y = - x^2$.
- understand the effect of translation and scaling on the graph of $y = x^2$

- rewrite a quadratic expression by completing the square
- use the rewritten form to sketch the graph of the general expression $ax^2 + bx + c$
- determine the highest or lowest point on the graph $y = ax^2 + bx + c$
- sketch the graph of a quadratic expression
- state the criterion that determines the number of roots of a quadratic equation
- solve the equation $ax^2 + bx + c = 0$ via factorisation, by completing the square and by the formula
- recognise the graphs on $y = x^3$ and $y = - x^3$.
- understand the effect of translation on the graph $y = x^3$
- state and use the remainder theorem
- factorise simple polynomials as a product of linear and quadratic factors

**2 Calculus**

Functions and their inverses

The student should be able to

- define a function, its domain and its range
- determine the domain and the range of a simple function
- draw a graph of a simple function
- determine whether a function is injective, surjective, bijective
- understand how a graphical translation can alter a functional description
- understand how a reflection in either axis can alter a functional description
- understand how a scaling transformation can alter a functional description
- determine the domain and the range of simple composite functions
- use appropriate software to plot the graph of a function
- determine the domain and the range of the inverse of a function
- obtain the inverse of a function graphically and algebraically
- determine any restrictions on $f(x)$ for the inverse to be a function
- obtain the inverse of a composite function
- recognise the properties of the function $1/x$

Sequences, series, binomial expansions

The student should be able to

- define a sequence and a series and distinguish between them
- recognise an arithmetic progression and its component parts
- find the general term of an arithmetic progression
- find the sum of the first $n$ terms of an arithmetic progression
- recognise a geometric progression and its component parts
- find the general term of a geometric progression
- find the sum of the first $n$ terms of a geometric progression
- interpret the term “sum” in relation to an infinite geometric series
- find the sum of an infinite geometric series when it exists
- find the arithmetic mean of two numbers
- find the geometric mean of two numbers
- obtain the binomial expansions of $(a + b)^s$, $(1+ x)^s$ for $s$ a rational number
- use the binomial expansion to obtain approximations to simple rational functions
- understand the concept of the limit of a function

Logarithmic and exponential functions

The student should be able to

- recognise the graphs of the power law function
- define the exponential function and sketch its graph
- define the logarithmic function as the inverse of the exponential function
- use the laws of logarithms to simplify expressions
- solve equations involving exponential and logarithmic functions
- solve problems using growth and decay models

Rates of change and differentiation

The student should be able to

- define average and instantaneous rates of change of a function
- understand how the derivative of a function at a point is defined
- recognise the derivative of a function as the instantaneous rate of change
- interpret the derivative as the gradient at a point on a graph
- distinguish between “derivative” and “derived function”
- use a table of the derived functions of simple functions
- recall the derived function of each of the standard functions
- use the multiple, sum, product and quotient rules
- use the chain rule
- relate the derivative of a function to the gradient of a tangent to its graph
- obtain the equation of the tangent and normal to the graph of a function

Stationary points, maximum and minimum values

The student should be able to

- use the derived function to find where a simple function is increasing or decreasing
- define a stationary point of a function
- distinguish between a turning point and a stationary point
- locate a turning point using the first derivative of a simple function
- classify turning points using first derivatives
- calculate the second derivative
- use the second derivative test

Indefinite integration

The student should be able to

- reverse the process of differentiation to obtain an indefinite integral for simple functions
- understand the role of the arbitrary constant
- use a table of indefinite integrals of simple functions
- understand and use the notation for indefinite integrals
- use the constant multiple rule and the sum rule
- use indefinite integration to solve practical problems such as obtaining velocity from a formula for acceleration or displacement from a formula for velocity

Definite integration, applications to areas and volumes

The student should be able to

- understand the idea of a definite integral as the limit of a sum
- realise the importance of the Fundamental Theorem of the Calculus
- obtain definite integrals of simple functions
- use the main properties of definite integrals
- calculate the area under a graph and recognise the meaning of a negative value
- calculate the are between two curves
- calculate the volume of a solid of revolution
- use trapezium and Simpson's rules to approximate the value of a definite integral

Complex numbers

The student should be able to

- define a complex number and identify its component parts
- represent a complex number on an Argand diagram
- carry out the operations of addition and subtraction of complex numbers
- write down the conjugate of a complex number and represent it graphically
- identify the modulus and argument of a complex number
- carry out the operations of multiplication and division in both Cartesian and polar form
- state and use the Euler's formula
- state and understand De Moivre's theorem for a rational index
- solve equations of the form $z^n = a,$ where $a$ is a real number
- find roots of a complex number
- describe regions in the plane by restricting the modulus and/or the argument of a complex number

## 3. Discrete Mathematics

Sets

The student should be able to

- understand the concepts of a set, a subset and the empty set
- determine whether an item belongs to a given set or not
- use and interpret Venn diagrams
- find the union and the intersection of two given sets
- apply the laws of set algebra

Proof

The student should be able to

- distinguish between an axiom and a theorem
- understand how a theorem is deduced from a set of axioms
- appreciate how a corollary is developed from a theorem
- follow a proof of one elementary theorem, e.g. Pythagoras theorem

## 4. Geometry and trigonometry

Geometry

The student should be able to

- recognise the different types of angle
- identify the equal angles produced by a transversal cutting parallel lines
- identify the different types of a triangle
- state and use the formula for the sum of the interior angles of a polygon
- calculate the area of a triangle
- use the rules for identifying congruent triangles
- know when two triangles are similar
- state and use Pythagoras' theorem
- understand radian measure
- convert from degrees to radians and vice-versa
- state and use the formulae for the circumference of a circle and the area of a disc
- calculate the length of a circular arc
- calculate the areas of a sector and of a segment of a circle
- quote formulae for the area of simple plane figures
- quote formulae for the volume of elementary solids: a cylinder, a pyramid, a tetrahedron, a cone and a sphere
- quote formulae for the surface area of elementary solids: a cylinder, a pyramid, a tetrahedron, a cone and a sphere
- sketch simple orthographic views of elementary solids
- understand the basic concept of a geometric transformation in the plane
- recognise examples of a metric transformation (isometry) and affine transformation (similitude)
- obtain the image of a plane figure in a defined geometric transformation: a translation in a given direction, a rotation about a given centre, a symmetry with respect to the centre or to the axis, scaling to a centre by a given ration

Trigonometry

The student should be able to

- define the sime, cosine, tangent and cotangent of an acute angle
- state and use the fundamental identities arising from Pythagoras' theorem
- relate the trigonometric ratios of an angle to those of its complement
- relate the trigonometric rations of an angle to those of its supplement
- state in which quadrants each trigonometric ratio is positive
- state and apply the sine rule
- state and apply the cosine rule
- calculate the area of a triangle from the lengths of two sides and the included angle
- solve a triangle given sufficient information about its sides and angles
- recognise when there is no triangle possible and when two triangle can be found

Co-ordinate geometry

The student should be able to

- calculate the distance between two points
- find the position of a point which divides a line segment in a given ration
- find the angle between two straight lines

- calculate the distance of a given point from a given line
- calculate the area of a triangle knowing the co-ordinates of its vertices
- give simple examples of a locus
- recognise and interpret the equation of a circle in standard form and state its radius and centre
- convert the general equation of a circle to standard form
- recognise the parametric equations of a circle
- derive the main properties of a circle, including the equation of the tangent at a point
- define a parabola as a locus
- recognise and interpret the equation of a parabola in standard form and state its vertex, focus, axis, parameter and directrix
- recognise the parametric equation of a parabola
- derive the main properties of a parabola, including the equation of the tangent at a point
- understand the concept of parametric representation of a curve
- use polar co-ordinates and convert to and from Cartesian co-ordinates

Trigonometric functions and application

The student should be able to

- define the term of a periodic function
- sketch the graphs of $\sin x$ , $\cos x$, $\tan x$ and $\cot x$ and describe their main features
- deduce the nature of the graphs of $a \sin x$, $a \cos x$ , $a \tan x$ and $a \cot x$
- deduce the nature of the graphs of $\sin ax$, $\cos ax$ , $\tan ax$ and $\cot ax$
- deduce the nature of the graphs of $\sin(x + a)$, $a + \sin x$, etc
- solve the equations $\sin x = c$, $\cos x = c$, $\tan x = c$, $\cot x = c$
- use the expression $a \sin(\omega t + \varphi)$ to represent an oscillation and relate the parameters to the motion
- rewrite the expression $a \cos \omega t + b \sin \omega t$ as a single cosine or sine formula

Trigonometric identities

The student should be able to

- obtain and use the compound angle formulae and double angle formulae
- obtain and use the product formulae
- solve simple problems using these identities

## 5. Statistics and probability

Data handling

The student should be able to

- interpret data presented in the form of line diagrams, bar charts, pie charts
- interpret data presented in the form of stem and leaf diagrams, box plots, histograms
- construct line diagrams, bar charts, pie charts, stem and leaf diagrams, box plots, histograms for suitable data sets
- calculate the mode, median and mean for a set of data items

Probability

The student should be able to

- define the terms “outcome”, “event” and “probability”
- calculate the probability of an event by counting outcomes
- calculate the probability of the complement of an event
- calculate the probability of the union of two mutually-exclusive events
- calculate the probability of the union of two events
- calculate the probability of the intersection of two independent events

Combinatorics

The student should be able to

- evaluate the number of ways of arranging unlike objects in a line
- evaluate the number of ways of arranging objects in a line, where some are alike
- evaluate the number of ways of arranging unlike objects in a ring
- evaluate the number of ways of permuting $r$ objects from $n$ unlike objects
- evaluate the number of combinations of $r$ objects from $n$ unlike objects
- use the multiplication principle for combinations

## Mathematics in Bachelor programmes

The material “MATHEMATICS FOR THE EUROPEAN ENGINEER, A Curriculum for the Twenty-first Century” [1] specified so called Core level 1 which contained the material to be covered by all engineering students. With the introduction of Bachelor programmes (which take 3 or 4 years - 6 to 8 semester) and with the reduction of mathematical education of bachelors, it is too ambitious and cannot be achieved. Our suggestion is to divide the material into two parts: The Part 1 contains the essentials that any bachelor in any technical programme should cover, the Part 2 brings topics from which different branches of engineering will choose the ones that are essential for them.

## Part 1

### 1. Analysis and Calculus

Functions of one variable – learning outcomes

The student should be able to

- define and recognise an odd function and an even function
- understand the properties “concave” and “convex” function
- identify, from its graph where a function is concave and where it is convex
- define and locate points of inflection on the graph of a function
- define and sketch the functions sinh, cosh, tanh and coth
- state the domain and range of the inverse hyperbolic functions
- recognise and use basic hyperbolic identities
- apply the functions to practical problems (for example, a suspended cable)
- sketch the graph of a rational function where the numerator is a linear expression and the denominator is either a linear expression or the product of two linear expressions
- obtain the partial fractions of a rational function, including cases where the denominator has a repeated linear factor or an irreducible quadratic factor.

Differentiation

The student should be able to

- understand the concepts of continuity and smoothness
- differentiate inverse functions
- differentiate functions defined implicitly
- differentiate functions defined parametrically
- locate any points of inflection of a function
- find greatest and least values of physical quantities

Sequences and series

The student should be able to

- understand convergence and divergence of a sequence
- understand the concept of a power series
- apply simple tests for convergence of a series
- find the tangent and quadratic approximations of a function
- understand the idea of radius of a convergence of a power series
- recognise Maclaurin series for standard functions
- understand how Maclaurin series generalise to Taylor series
- use Taylor series to obtain approximate percentage changes in a function

Methods of integration

The student should be able to

- obtain definite and indefinite integrals of rational functions in partial fraction form
- apply the method of integration by parts to indefinite and definite integrals
- use the method of substitution on indefinite and definite integrals
- solve practical problems which require the evaluation of an integral
- recognise simple examples of improper integrals
- use te formula for maximum error in a trapezoidal rule estimate
- use the formula for the maximum error in a Simpson's rule estimate

Application of integration

The student should be able to

- find the length of part of a plane curve
- find the curved surface area of a solid of revolution
- obtain the mean value and root-mean-square (RMS) value of a function in a closed interval
- find the first and second moments of a plane area about an axis

Solution of non-linear equations

The student should be able to

- use intersecting graphs to help locate approximately the roots of non-linear equations
- use Descartes' rules of signs for polynomial equations
- understand the distinction between point estimation and interval reduction methods
- use a point estimation method and an interval reduction method to solve a practical problem
- understand various convergence criteria
- use appropriate software to solve non-linear equations

Functions of two or more variables

The student should be able to
- evaluate a function of two or more variable at a given point
- understand the concepts of continuity and smoothness
- relate the main features, including stationary points, of a function of 2 variables to its 3D plot and to a contour map
- obtain the first partial derivatives of simple functions of several variables
- obtain the directional derivatives and gradient vector for a function of two or more variables
- obtain tangent plane and normal line of a simple function of two variables
- use appropriate software to produce 3D plats and/or contour maps

Ordinary differential equations

The student should be able to
- understand how rates of change can be modelled using first and second derivatives
- recognise the kinds of boundary condition which apply in particular situations
- distinguish between boundary and initial conditions
- distinguish between general solution and particular solutions
- understand how existence and uniqueness relate to a solution
- classify differential equations and recognise the nature of their general solution

## 2. Linear algebra

Vector algebra and applications

The student should be able to
- define a scalar product of two vectors and use it in simple applications
- understand the geometric interpretation of the scalar product
- define the vector product of two vectors and use it in simple applications
- understand the geometric interpretation of the vector product
- define the scalar triple product of three vectors and use it in simple applications
- understand the geometric interpretation of the scalar triple product

Matrices and determinants

The student should be able to
- understand what is meant by a matrix
- recall the basic terms associated with matrices (e.g. diagonal, trace, square, triangular identity)
- obtain the transpose of a matrix
- determine any scalar multiple of a matrix
- recognise when two matrices can be added and find, where possible, their sum
- recognise when two matrices can be multiplied and find, where possible, their product
- understand the relation between elementary row (column) operations and multiplication of matrices
- calculate the determinant of 2 x 2 and 3 x 3 matrices
- use the elementary properties of determinants in their calculation
- calculate the determinant of $n$ x $n$ matrix using row (column) expansion

- state a criterion for a square matrix to have an inverse
- write down the inverse of a 2 x 2 matrix when it exists
- determine the inverse of a matrix, when it exists, using row operations
- calculate the rank of a matrix
- use appropriate software to determine inverse matrices

Solution of simultaneous linear equations

The student should be able to
- represent a system of linear equations in matrix form
- understand how the general solution of an inhomogeneous linear system of $m$ equations in $n$ unknowns is obtained from the solution of the homogeneous system and a particular solution
- recognise the different possibilities for the solution of a system of linear equations
- give a geometrical interpretation of the solution of a system of linear equations
- understand how and why the rank of the coefficient matrix and the augmented matrix of a linear system can be used to analyse its solutions
- use the inverse matrix to find the solution of 3 simultaneous linear equations
- understand the term "ill-conditioned"
- apply the Gauss elimination method and recognise when it fails
- understand the Gauss-Jordan variation
- use appropriate software to solve simultaneous linear equations

Linear spaces and transformations

The student should be able to
- define a linear space
- define and recognise linear independence
- define and obtain a basis for a linear space
- define a subspace of a linear space and find a basis for it
- define scalar product in a linear space
- define the Euclidean norm
- define a linear transformation between two space; define the image space and the null space for the transformation
- derive the matrix representation of a linear transformation
- understand how to carry out a change of basis
- define an orthogonal transformation
- apply the above matrices of linear transformation in the Euclidean plane and Euclidean space
- recognise matrices of Euclidean and affine transformation: identity, translation, symmetry, rotation and scaling

### 3. Statistics and probability

Data handling

The student should be able to
- calculate the range, inter-quartile range, variance and standard deviation for a set of data items
- distinguish between a population and a sample
- know the difference between the characteristic values (moments) of a population and a sample
- construct a suitable frequency distribution from a data set
- calculate relative frequencies

- calculate measures of average and dispersion for a grouped set of data
- understand the effect of grouping on these measures

Simple probability

The student should be able to
- interpret probability as a degree of belief
- understand the distinction between *a priori* and *a posteriori* probabilities
- use a tree diagram to calculate probabilities
- know what conditional probability is and be able to use it (Bayes' theorem)
- calculate probabilities for series and parallel connections

Probability models

The student should be able to
- define a random variable and a discrete probability distribution
- state the criteria for a binomial model and define its parameters
- calculate probabilities for a binomial model
- state the criteria for a Poisson model and define its parameters
- calculate probabilities for a Poisson model
- state the expected value and variance for each of these models
- understand when a random variable is continuous
- explain the way in which probability calculations are carried out in the continuous case

Normal distribution

The student should be able to
- handle probability statements involving continuous random variables
- convert a problem involving a normal variable to the area under part of its density curve
- relate the general normal distribution to the standardised normal distribution
- use tables for the standardised normal variable
- solve problems involving a normal variable using tables

Sampling

The student should be able to
- define a random sample
- know what a sampling distribution is
- understand the term "mean squared error" of an estimate
- understand the term "unbiasedness" of an estimate]

Statistical inference

The student should be able to
- apply confidence intervals to sample estimates
- follow the main steps in a test of hypothesis
- understand the difference between a test of hypothesis and a significance test (p-value)
- define the level of a test (error of the first kind)
- define the power of a test (error of the second kind)

- state the link between the distribution of a normal variable and that of means of samples
- place confidence intervals around the sample estimate of a population mean
- test claims about the population mean using results from sampling
- recognise whether an alternative hypothesis leads to a one-tail or a two-tail test
- compare the approaches of using confidence intervals and hypothesis tests

## Part 2

### 1. Analysis and Calculus

First order ordinary differential equations

The student should be able to
- recognise when an equation can be solved by separating its variables
- obtain general solutions to equations by applying the method
- obtain particular solutions by applying boundary or initial conditions
- recognise the equations of the main areas of application
- interpret the solution and its constituent parts in terms of the physical problem
- understand the term “exact equation”
- obtain the general solution of an exact equation
- understand the purpose of an integrating factor
- obtain the general solution of an exact equation
- use the integrating factor to solve such equations
- find and interpret solutions to equations describing standard physical situations
- use a simple numerical method for estimating points of the solution curve

Second order equations – complementary function and particular integral

The student should be able to
- distinguish between free and forced oscillation
- recognise linear second-order equations with constant coefficients and how they arise in the modelling of oscillation
- obtain the auxiliary equation
- obtain the types of complementary function and interpret them in terms of the model
- find the particular integral for the simple forcing functions
- obtain the general solution to the equation of motion
- apply the initial conditions to obtain a particular solution
- identify the transient and steady-state response
- recognise and understand the meaning of “beats”
- recognise and understand the meaning of resonance

Functions of several variables

The student should be able to
- obtain the first partial derivatives of simple functions of several variables
- define a stationary point of a function of several variables
- define local maximum, local minimum and saddle point for a function of two variables
- locate the stationary points of a function of several variables
- understand the criteria for classifying a stationary point
- obtain total rates of change of functions of several variables

- approximate small errors in a function using partial derivatives

Least squares curve fitting

The student should be able to

- define the least squares criterion for fitting a straight line to a set of data points
- use the normal equations to find the gradient and intercept of this line
- understand how to modify the method to deal with polynomial models
- recognise the role of partial differentiation in the process

Solution of non-linear equations

The student should be able to

- use intersection graphs to help locate approximately the roots of non-linear equations
- use Descartes' rules of signs for polynomial equations
- understand the distinction between point estimation and interval reduction methods
- understand and use the method of successive bisection
- understand and use the method of false position
- understand and use the Newton-Raphson method
- understand and use the method of basic (fixed-point) iteration
- understand the various convergence criteria

Introduction to Fourier series

The student should be able to

- understand the effects of superimposing sinusoidal waves of different frequencies
- recognise that a Fourier series approximation can be derived by a least squares approach
- understand the idea of orthogonal functions
- use the formulae to find Fourier coefficients in simple cases
- appreciate the effect of including more terms in the approximation
- interpret the resulting series, particularly the constant term
- comment on the usefulness of the series obtained

Double integrals

The student should be able to

- interpret the components of a double integral
- sketch the area over which a double integral is defined
- evaluate a double integral by repeated integration
- reverse the order of a double integral
- convert a double integral to polar coordinates and evaluate it
- find volumes using double integrals

## 2. Discrete mathematics

Mathematical logic

The student should be able to

- recognise a proposition
- negate a proposition
- form a compound proposition using the connectives AND, OR, IMPLICATION
- construct truth table for a compound proposition
- construct a truth table for an implication
- verify the equivalence of two statements using a truth table
- identify a contradiction and a tautology
- construct the converse statement
- obtain the contrapositive form of an implication
- understand the universal quantifier “for all”
- understand the existential quantifier “there exists”
- negate propositions with quantifiers
- follow simple examples of direct and indirect proof
- follow a simple example of a proof by contradiction

Sets

The student should be able to

- understand the notion of an ordered pair
- find the Cartesian product of two sets
- define a characteristic function of a subset of a given universe
- compare the algebra of switching circuits to that of set algebra
- analyse simple logic circuits comprising AND, OR, NAND, NOR and EXCLUSIVE OR gates
- understand the concept of a countable set

Relations

The student should be able to

- understand the notion of a binary relation
- find the composition of two binary relations, when it exists
- find the inverse of a binary relation
- understand the notion of a ternary relation
- understand the notion of an equivalence relation on a set
- verify whether a given relation is an equivalence relation or not
- understand the notion of a partition on a set
- view an equivalence either as a relation or a partition
- understand the notion of a partial order on a set
- understand the difference between maximal and greatest element, and between minimal and smallest element

Mathematical induction and recursion

The student should be able to

- understand (weak) mathematical induction
- follow a simple proof which uses mathematical induction
- define a set by induction
- use structural induction to prove some simple properties of a set which is given by induction
- understand the concept of recursion

- define the factorial of a positive integer by recursion (any other suitable example will serve just as well)

Difference equations

The student should be able to
- define a sequence by a recursive formula
- obtain the general solution of a linear first-order difference equation with constant coefficients
- obtain the particular solutions of a linear first-order difference equation with constant coefficients which satisfies given conditions
- obtain the general solution of a linear second-order difference equation with constant coefficients
- obtain the particular solution of a linear second-order difference equation with constant coefficients which satisfies given conditions

Number systems

The student should be able to
- recognise the Peano axioms
- carry out arithmetic operations in the binary system
- carry out arithmetic operations in the hexadecimal system
- use Euclid's algorithm for finding the greatest common divisor

Algebraic operations

The student should be able to
- understand the notion of a group
- establish the congruence of two numbers modulo $n$
- understand and carry out arithmetic operations in $Z_n$, especially in $Z_2$
- carry out arithmetic operations on matrices over $Z_2$
- understand the Hamming code as an application of the above (any other suitable code will serve just as well)

Graphs

The student should be able to
- recognise a graph (directed and/or underictで) in a real situation
- understand the notions of a path and a cycle
- understand the notion of a tree and a binary tree
- recognise an Euler trail in a graph and/or an Euler graph
- recognise a Hamilton cycle (path) in a graph
- find components of connectivity in a graph
- find components of strong connectivity in a directed graph
- find a minimal spanning tree of a given graph

Algorithms

The student should be able to
- understand when an algorithm solves a problem

- understand the “big *O*” notation for functions
- understand the worst case analysis of an algorithm
- understand one of the sorting algorithms
- understand the idea of depth-first search
- understand the idea of breadth-first search
- understand a multi-stage algorithm (e.g. finding the shortest path, finding the maximal flow)

### 3. Geometry

Conic sections

The student should be able to
- recognise the equation of an ellipse in standard form and state its foci, semiaxes and directrices
- recognise the parametric equations of an ellipse
- derive the main properties of an ellipse, including the equation of the tangent at a point
- recognise the equation of a hyperbola in standard form and find its, foci, semiaxes and asymptotes
- recognise parametric equations of a hyperbola
- derive the main properties of a hyperbola, including the equation of the tangent at a point
- recognise the equation of a conic section in the general form and classify the type of conic section

3D co-ordinate geometry

The student should be able to
- recognise and use the standard equation of a straight line in 3D
- recognise and use the standard equation of a plane
- find the angle between two straight lines
- find where two straight lines intersect
- find the angle between two planes
- find the intersection line of two planes
- find the intersection of a line and a plane
- find the angle between a line and a plane
- calculate the distance between two points, a point and a line, a point and a plane
- calculate the distance between two lines, a line and a plane, two planes
- recognise and use the standard equation of a singular quadratic surface (cylindrical, conical)
- recognise and use the standard equation of a regular quadratic surface (ellipsoid, paraboloid, hyperboloid)

## Mathematics in Master Programmes

Mathematics education in Mater programmes should build on the foundations laid during the Bachelor programmes. This means that students accepted for Master programmes in engineering must have thorough knowledge of basic mathematical concepts. Also it is recommendable that at least part of the mathematical education should use standard mathematical software tools to enable students to solve problems from “real life”. It is also possible that during the study of a Master programme some topics omitting during the bachelor programme will be added. The extent of Mathematics education strongly depends on the branch of engineering.

1. **Analysis and Calculus**

Vector calculus

The student should be able to
- understand the concept of a vector field
- obtain the divergence of a vector field
- obtain the curl of a vector field
- apply properties of the operator “nabla”
- know that the curl of the gradient of a scalar is the zero vector
- know that the divergence of the curl of a vector is zero
- define and use the Laplacian operator

Line and surface integrals, integral theorems

The student should be able to
- evaluate line integrals along simple paths
- apply line integrals to calculate work done
- apply Green’s theorem in the plane to simple examples
- evaluate surface integrals over simple surfaces
- use the Jacobian to transform a problem into a new co-ordinate system
- apply Gauss’ divergence theorem to simple problems
- apply Stokes’ theorem to simple examples

Linear optimisation

The student should be able to
- recognise a linear programming problem in words and formulate it mathematically
- represent the feasible region graphically
- solve a maximisation problem graphically by superimposing lines of equal profit
- carry out simple sensitivity analysis
- represent and solve graphically a minimisation problem
- explain the term “redundant constraint”

The simplex method

The student should be able to
- convert a linear programming problem into a simplex tableau
- solve a maximisation problem by the simplex method
- interpret the tableau at each stage of the journey round the simplex
- recognise cases of failure
- write down the dual to a linear programming problem
- use the dual problem to solve a minimisation problem

Non-linear optimisation

The student should be able to

- solve an unconstrained optimisation problem in two variables
- use information in a physically-based problem to help obtain the solution
- use the method of Lagrange multipliers to solve constrained optimisation problems
- solve problems of minimising surface area for a fixed enclosed volume
- solve problems of minimising enclosed volume for a fixed surface area

Laplace transforms

The student should be able to

- use tables to find the Laplace transforms of simple functions
- use the property of linearity to find the Laplace transforms
- use the fist shift theorem to find the Laplace transforms
- use the "multiply by $t$" theorem to find the Laplace transforms
- obtain the transforms of first and second derivatives
- invert a transform using tables and partial fractions
- solve initial-value problems using Laplace transforms
- obtain the Laplace transform of a periodic function
- know the Laplace transform of the unit impulse function
- obtain the transfer function of a simple linear time-variant system
- obtain the impulse response of a simple system
- apply initial-value and final-value theorems
- obtain the frequency response of a simple system

z transforms

The student should be able to

- recognise the need to sample continuous-time functions to obtain a discrete-time signal
- obtain the z transforms of simple sequences
- use the linearity and shift properties to obtain z transforms
- know the "multiply by $a^K$" and "multiply by $k$" theorems
- use the initial-value and final-value theorems
- invert a transform using tables and partial fractions
- solve initial-value problems using z transforms
- compare this method of solution with the method Laplace transforms

Complex functions

The student should be able to

- define a complex function and an analytic function
- determine the image path of a linear mapping
- determine the image path under the inversion mapping
- determine the image path under a bilinear mapping
- determine the image path under the mapping $w = z^2$
- understand the concept of conformal mapping and know and apply some examples
- verify that a given function satisfies the Cauchy-Riemann conditions
- recognise when complex functions are multi-valued
- define a harmonic function
- find the conjugate to a given harmonic function

Complex series and contour integration

The student should be able to
- obtain the Taylor series of simple complex functions
- determine the radius of convergence of such series
- obtain the Laurent series of simple complex functions
- recognise the need for different series in different parts of the complex plane
- understand the terms “singularity”, pole
- find the residue of a complex function at a pole
- understand the concept of a contour integral
- evaluate a contour integral along simple linear paths
- use Cauchy’s theorem and Cauchy’s integral theorem
- state and use the residue theorem to evaluate definite real integrals

Introduction to partial differential equations

The student should be able to
- recognise the three main types of second-order linear partial differential equations
- appreciate in outline how each of these types is derived
- state suitable boundary conditions to accompany each type
- understand the nature of the solution of each type of equation

Solving partial differential equations

The student should be able to
- understand the main steps in the separation of variables method
- apply the method to the solution of Laplace’s equation
- interpret the solution in terms of the physical problem

## 2. Discrete mathematics

Graph Theory

The student should be able to
- understand the problem of finding shortest paths in a directed weighted graph
- find shortest paths from a given vertex to any other vertex in a given directed weighted graph without cycles of negative length
- find shortest paths between any two vertices of the given directed weighted graph without cycles of negative length
- understand the notion of an independent set in an undirected graph
- understand the notion of a clique in an undirected graph
- understand the notion of (vertex) colouring, and *k*-colourability
- decide whether a graph is 2-colourable
- understand the notion of matching
- understand the notion of a planar graph
- apply different graph theoretical notion in real life problems

Groups and generalisation

The student should be able to

- understand the notion of a binary operation on a set
- decide whether a given binary operation is associative, commutative, idempotent, whether it has a identity element
- understand the notion of a group
- decide whether a given set with a binary operation is a group
- decide whether a given subset of a group forms a subgroup
- work with finite groups
- define order of an element of a finite group
- solve equations of the form $x^n = a$ in a finite group, mainly in $Z_m$
- use the Euler-Fermat theorem

Rings and fields

The student should be able to

- work in $Z_m$ with operations addition and multiplication
- understand the notion of a ring
- understand the notion of a field
- decide where $Z_m$ is a field
- work in the ring of polynomials over the field $Z_p$
- work in the finite field $GF(p^k)$, where $p$ is a prime number

Lattices

The student should be able to

- draw a Hasse diagram of a Poset (partial order set)
- decide whether a Poset is a lattice
- define the operations of suprema and infima in a lattice given as a Poset
- decide whether a given lattice is a complete lattice
- decide whether a given lattice is distributive lattice
- decide whether a given element in a lattice with the smallest and greatest elements has a complement
- define a Boolean algebra and work in a Boolean algebra

## 3. Geometry

Helix

The student should be able to

- recognise the parametric equation of a helix
- derive the main properties of a helix, including the equation of the tangent at a point, slope and pitch

Geometric spaces and transformations

The student should be able to

- define Euclidean space and state its general properties
- understand the Cartesian co-ordinate system in the space
- apply the Euler transformations of the co-ordinate system

- understand the polar co-ordinate system in the plane
- understand the cylindrical co-ordinate system in the space
- understand the spherical co-ordinate system in the space
- define affine space and state its general properties
- understand the general concept of a geometric transformation on a set of points
- understand the terms “invariants” and “invariant properties”
- know and use the non-commutativity of the composition of transformations
- understand the group representation of geometric transformations
- classify specific groups of geometric transformations with respect to invariants
- derive the matrix form of basic Euclidean transformations
- derive the matrix form of an affine transformation
- calculate coordinates of an image of a point in a geometric transformation
- apply a geometric transformation to a plane figure

## 4. Linear algebra

Matrix methods

The student should be able to

- define a banded matrix
- recognise the notation for a tri-diagonal matrix
- use the Thomas algorithm for solving a system of equations with a tri-diagonal coefficient matrix
- carry out addition and multiplication of suitably-partitioned matrices
- find the inverse of a matrix in partitioned form

Eigenvalue problems

The student should be able to

- interpret eigenvectors and eigenvalues of a matrix in terms of the transformation it represents
- convert a transformation into a matrix eigenvalue problem
- find the eigenvalues and eigenvectors of 2 x 2 and 3 x 3 matrices algebraically
- determine the modal matrix for a given matrix
- reduce a matrix to diagonal form
- reduce a matrix to Jordan form
- state the Cayley-Hamilton theorem and use it to find powers and the inverse of a matrix
- understand a simple numerical method for finding the eigenvectors of a matrix
- use appropriate software to compute the eigenvalues and eigenvectors of a matrix
- apply eigenvalues and eigenvectors to the solution of systems of linear difference and differential equations
- understand how a problem in oscillatory motion can lead to an eigenvalue problem
- interpret the eigenvalues and eigenvectors in terms of the motion
- define a quadratic form and determine its nature using eigenvalues

## 5. Statictics and Probability

One-dimensional random variables

The student should be able to

- compare empirical and theoretical distributions

- apply the exponential distribution to simple problems
- apply the normal distribution to simple problems
- apply the Weibull distribution to simple problems
- apply the gamma distribution to simple problems

Two-dimensional random variables

The student should be able to
- understand the concept of a joint distribution
- understand the terms "joint density function", "marginal distribution functions"
- define independence of two random variables
- solve problems involving linear combinations of random variables
- determine the covariance of two random variables
- determine the correlation of two random variables

Small sample statistics

The student should be able to
- realise that the normal distribution is not reliable when used with small samples
- use tables of the t-distribution
- solve problems involving small-sample means using the t-distribution
- use tables of the F-distribution
- use pooling of variances where appropriate
- use the method of pairing where appropriate

Small sample statistics: chi-square tests

The student should be able to
- use tables for chi-squared distribution
- decide on the number of degrees of freedom appropriate to a particular problem
- use the chi-square distribution in tests of independence (contingency tables)
- use the chi-square distribution in tests of goodness of fit

Analysis of variance

The student should be able to
- set up the information for a one-way analysis of variance
- interpret the ANOVA table
- solve a problem using one-way analysis of variance
- set up the information for a two-way analysis of variance

Simple linear regression

The student should be able to
- derive the equation of the line of best fit to a set of data pairs
- calculate the correlation coefficient
- place confidence intervals around the estimates of slope and intercept
- place confidence intervals around values estimated from the regression line

- carry out an analysis of variance to test goodness of fit of the regression line
- interpret the results of the tests in terms of the original data
- describe the relationship between linear regression and least squares fitting

Multiple linear regression and design of experiments

The student should be able to
- understand the ideas involved in a multiple regression analysis
- appreciate the importance of experimental design
- recognise simple statistical designs

# Appendix

## Comments on the document "Mathematics Education for Engineers in the Changing World" – by the University of Minho, Portugal

Prof. Rosa Vasconcelos, University of Minho, Portugal

We would like to contribute to the discussion of the document by sharing some of our practices regarding mathematics in engineering education.

We acknowledge the importance of assessment and would like to emphasise the value of formative assessment combined with more involvement of students in the assessment process. In our opinion, students need to experience that they are not only part of their learning process, they are responsible for their own learning process. Student themselves control how they learn (deep or superficial) and what they learn. This responsibility for the learning process implies a responsibility for the assessment process as well, because the way students are assessed determines to a great extent the way students learn. Involving them in their assessment process and making them responsible, partly, for this process helps to increase their responsibility. Students, who define assessment criteria, correct and grade their work or that of their peers and write feedback, are more likely to regard the assessment process as fair and clear. They start to understand how they are assessed and recognise assessment results as a consequence of their own performance, or, as we can also say, they do no longer "blame the teacher" if they fail. Student involvement in (formative) assessment helps them to become more responsible for their own learning, as is also required by the Bologna Declaration. They need to be lifelong learners who reflect on their own learning and can steer their learning processes.

Our experiences with student involvement in assessment show us that students change their learning approach when they are required to participate actively in their assessment. They have proven to be able to share responsibilities in the assessment process with their teachers and can correct and grade themselves and their peers accurately. We find this involvement very valuable and would like to emphasise ongoing attention for student involvement in continuing formative assessment.

Apart from experiences with student involvement, our engineering courses are transforming more and more into project led courses in which traditional education in a number of subjects per semester is replaced by one project in which competencies of all subject are integrated. The theme of the project and the problem that is posed to the students requires the development of competencies in various areas. In the first pilot project mathematics subjects were not integrated in the project, because it was complicated for the mathematics teachers to imagine how the learning outcomes of their courses could integrate in the project. As a result, students worked on one large project that included four or five subjects and apart from that had to work on the mathematics subjects. This led to an unbalanced workload and performance of students. They felt that the mathematics subject was not really part of their curriculum and the teaching staff was not happy with the low attendance of classes. In current project led education experience at the University of Minho, mathematics subjects are an integral part of the project. This has a number of advantages for both students and teaching staff.

Firstly, mathematics becomes part of the regular working rhythm of the students and is not experienced as a subject with a different status. Secondly, as it becomes part of the project, it needs to be well worked out in what way the content of a certain mathematics subject can make sense in the project theme. The theme and the problem to solve in the project are designed in such a way that a certain level of mathematical competence is required to develop a working solution. Designing such a problem is requires a team of dedicated teachers who are prepared to look over the boundaries of their own subject and can reflect critically on the viability of a project.

Once a project is designed, students experience the integration of different subject areas and construct knowledge on different issues in a practical, real-life context. Learning by projects adds a meaningful context to the different subjects. Especially in mathematics, this is important, because students start to see the usefulness of mathematics in engineering projects. Integrating mathematics in projects helps students to realise that they really need to knowledge and skills form the mathematics subjects to solve certain engineering problems. Furthermore, the design and subsequent coordination for the project by a project team promotes the exchange of information between teachers on their subjects and softens the sometimes artificial border between subject areas.

In our opinion, this primary positive and applied experience with mathematics, early in an engineering course, can serve as a motivation for more advanced mathematics in later years. The basic skills are fundamental, and need to be incorporated well in an early stage. Developing conceptual thinking through more abstract material will probably face less resistance from students, once they have developed a positive attitude towards mathematics through real integration of mathematics in engineering problems.

## Refernces

[1] MATHEMATICS FOR THE EUROPEAN ENGINEER, A Curriculum for the Twenty-first Century, Mustoe, L R and Lawson, D A (Eds), (2002), published by SEFI HQ, ISBN 2-87352-045-0

[2] Lawson, D A: Assessment in Engineering Mathematics; Proceedings of 12th SEFI Mathematics Working Group Seminar, Vienna, 2002, pp 34 – 41

[3] Alpers, B: The mathematical expertise of mechanical engineers, Proceedings of 13th SEFI Mathematics Working Group Seminar, Kongsberg, 2006, pp 11-17