European Society for Engineering Education (SEFI)

A Framework for Mathematics Curricula in Engineering Education

A Report of the Mathematics Working Group

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Contents

Executive Summary

It is the goal of SEFI's Mathematics Working Group to provide a discussion forum and orientation to those who are interested in the mathematical education of engineering students in Europe. An important contribution to this goal is the group's curriculum document which was first issued in 1992. After ten years, in 2002, a second edition was published which brought the document more in line with current curriculum practices by formulating a detailed and structured list of concrete content-related learning outcomes. During the last decade, in many seminars of the group the topic of higher-level learning goals and outcomes came up. It is the intention of the current third edition of the curriculum document to state, explain and exemplify a framework for systematically including such higher-level learning goals based on state-of-the-art educational research. For this, the competence concept developed in Denmark and later adopted in the famous OECD PISA study, is used. Mathematical competence is the ability to recognize, use and apply mathematical concepts in relevant contexts and situations which certainly is the predominant goal of the mathematical education for engineers. Therefore, the main message of this new edition is that although contents are still important, they should be embedded in a broader view of mathematical competencies.

This document adapts the competence concept to the mathematical education of engineers and explains and illustrates it by giving examples. It also provides information for specifying the extent to which a competency should be acquired. It **does not** prescribe a particular level of progress for competence acquisition in engineering education. There are many different engineering branches and many different job profiles with various needs for mathematical competencies such that it does not make sense to specify a fixed profile. The competence framework serves as an analytical framework for thinking about the current state in one's own institution and also as a design framework for specifying the intended profile. A sketch of an example profile for a practice-oriented study course in mechanical engineering is given in the document. The document retained the slightly changed list of content-related learning outcomes that formed the "kernel" of the previous curriculum document. These are still important since lecturers teaching application subjects want to be sure that students have at least an "initial familiarity" with certain mathematical concepts and procedures they need in their application modelling. In order to provide sense-making beyond the purely mathematical structure, overarching themes like "measuring" or "functional dependency" were identified as was also done in the OECD PISA document.

In order to offer helpful orientation for designing teaching processes, teaching and learning environments are outlined which help students to obtain the competencies to an adequate degree. It is clear that such competencies cannot be obtained by just listening to lectures, so adequate forms of active involvement of students need to be installed. Topics like use of technology and integration of mathematics and engineering education are also discussed. Since assessment procedures determine to a good extent the behaviour of students and are hence important for really achieving progress in competencies, different forms of assessment which are adequate for capturing certain kinds of achievements are discussed.

The main purpose of this document is to provide orientation for those who set up concrete mathematics curricula for their specific engineering programme, and for lecturers who think about learning and assessment arrangements for achieving the intended level of competence acquisition. It also serves as a framework for the group's future work and discussions.

“… those who are developing new curricula should,
despite reformist zeal, proceed with due caution.”
Alan H. Schoenfeld (1994)

1 Introduction – Goals and Use of the Curriculum

When the SEFI Mathematics Working Group set up its first “Core Curriculum” in 1992, Peter Nüesch, one of the co-authors and former SEFI president, wrote in his address preceding the curriculum (Barry & Steele 1992, p.8): “It is hoped that our Core Curriculum answers only the one very essential question: what should be the content of mathematics courses for engineers?” Accordingly, the “heart” of the curriculum document consisted of a list of topics to be dealt with, organized on different levels, although it is fair to state that other issues concerning the educational process were briefly commented on. For the second edition of the curriculum document in 2002 (Mustoe & Lawson 2002), one motivation for change was to bring the curriculum more in line with current curriculum practices and “… phrase a curriculum in terms of learning outcomes rather than a list of topics to be covered” (p.2). This resulted in a quite detailed organized list of content-related learning outcomes. Moreover, other issues like the role of technology, transition problems and other educational goals like communication and modelling were included in a short commentary section.

During the last decade, in many seminars of the group the topic of higher-level learning goals and outcomes came up. This can be found specifically in the contribution by (Booth 2004) on “learning for understanding” and the paper by (Cardella 2008) on using a “broad notion of mathematical thinking”. Although the curriculum document as of 2002 contains some short statements on such goals (chapter 4, p.47), it does not apply a systematic approach which could provide a framework for other didactical issues in the document. It is the intention of the current third edition of the curriculum document to state, explain and exemplify such a framework based on state-of-the-art educational research. Nevertheless, contents and content-related learning outcomes still provide important orientation for what colleagues in application subjects expect from the mathematical education of engineers. Therefore, the main message of this new edition is that although contents are still important, they should be embedded in a broader view of mathematical competencies that the mathematical education of engineers strives to achieve. The history of the curriculum document so far can hence be described as going “from contents to outcomes to competencies”.

When trying to set up a framework for specifying higher-level goals based on current insights from educational research, there are several sources available within the general mathematics education community aiming at school mathematics or undergraduate education or both (for an overview of curricular trends in tertiary education see (Hillel 2001)). Cardella (2008) proposes to use the aspects of mathematical thinking identified by Schoenfeld (1992, 1994) to broaden the view of what mathematical education of engineers should strive for. Schoenfeld emphasizes that beside content knowledge, there are problem solving strategies, meta-cognitive processes in using resources, beliefs and affects and mathematical practices which together make up mathematical thinking:

> *“… mathematical thinking consists of a lot more than knowing facts, theorems, techniques, etc. … I would characterize the mathematics a person understands by describing what that person can do mathematically, rather than by an inventory of what the person ‘knows.’” (Schoenfeld 1994)*

Schoenfeld's aspects can also be found when observing engineering students as well as engineers working on practical tasks (Cardella 2008). Asiala et al. (1996) similarly present a broad perspective of "what it means to learn and know something in mathematics":

> *"An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations".*

In 2004, the Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America wrote a "Curriculum Guide" which contains recommendations that argue along similar lines (Barker et al. 2004). Among other items, the recommendations state:

> *"… Every course should incorporate activities that will help all students progress in developing analytical, critical reasoning, problem-solving, and communication skills and acquiring mathematical habits of mind. … Promote awareness of connections to other subjects …. And strengthen each student's ability to apply the course material to these subjects. … At every level of the curriculum, some courses should incorporate activities that will help all students progress in learning to use technology …" (p.1 and p.2)*

The report is also based on several workshops where members of "partner disciplines" (including engineering) stated their understanding of the mathematical qualifications needed for being successful in the discipline (Ganter & Barker 2004).

Finally, in the Danish KOM project a group headed by Niss organized their description of what mathematical education intends to achieve around the notion of competence which also strongly influenced the description of educational goals in the famous OECD-PISA study (OECD 2009):

> *"<u>Mathematical competence</u> (in the original italics are used instead of underlining) then means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. Necessary, but certainly not sufficient, prerequisites for mathematical competence are lots of factual knowledge and technical skills …" (Niss 2003a, p.6/7)*

Blomhoj and Jensen (2007, p. 47) put it in a nutshell by defining a mathematical competency (an ingredient of mathematical competence) as

> *"… someone's insightful readiness to act in response to <u>a certain kind of mathematical challenge</u> (in the original italics are used instead of underlining) of a given situation …".*

In order to be useful, the KOM project identified a list of such mathematical competencies which overlap but have different emphasis: thinking mathematically; posing and solving mathematical problems; modelling mathematically; reasoning mathematically; representing mathematical entities; handling mathematical symbols and formalism; communicating in, with, and about mathematics; making use of aids and tools. This list is explained in more detailed in (Niss 2003a,b). It is meant to be a framework (like the aspects stated by Schoenfeld) overarching all stages of education including tertiary education. For a certain educational setting like engineering education, the specific "mathematical challenges" have to be identified and the competencies must be interpreted in this context. In order to describe progress in obtaining the competencies during different stages of education the KOM project identified three dimensions (degree of coverage, radius of action, technical level, see (Niss 2003a, p. 10)). These can be used to analyze or prescribe in more detail what one wants the students to achieve at a certain level of education or for a certain educational profile.

Even if the above descriptions provide just a short glimpse into the concepts the authors use, they show the large degree of commonality in identifying a broader spectrum of goals mathematical education should strive for, going far beyond a content-based approach. In this curriculum, we use the competence-based framework set up in the Danish KOM project to identify the higher-level goals of the mathematical part of engineering education. This is also in line with current trends in general engineering education where the notion of "competence" has been used to describe educational activities which favour "action-based knowledge over knowledge simply held, in the name of performance and effectiveness" (Lemaitre et al. 2006, p.47). Competence in this sense is contextualized, i.e. related to a "field of activity, a series of specific tasks … and a given situation" (p.50). However, it should be noted that the term "competence" is used very differently by various authors (even including the meaning of lower-level skill) and that on the other hand other terms like "skill", "capability", "capacity" are used in literature with a meaning similar to the one given for "competence" (for a discussion of the confusing usage of these terms see Lemaitre et al. (2006)).

The second chapter of this document describes the eight competencies and the three dimensions of progress in more detail. Moreover, we give illustrative engineering mathematics examples for the competencies. We do not prescribe a particular level of progress for engineering education. There are many different engineering branches and many different job profiles with various needs for mathematical competencies such that it does not make sense to specify a fixed profile. On the surface, this would facilitate student exchange but it would neglect the difference and hence would have a low probability of being used. The competence framework can serve as an analytical framework for thinking about the current state in one's own institution and also as a design framework for specifying the intended profile. Identifying such a profile might even help to improve exchange which is not based on just one unique profile which every institution subscribes to but hardly anyone realizes. It can enable exchange between those institutions which have comparable profiles such that an exchange is easily possible. The final section in chapter 2 sketches how such a profile could look like for a practice-oriented study course in mechanical engineering.

The third chapter deals with content-related competencies as well as learning outcomes concerning knowledge and skills. The latter formed the "kernel" of the curriculum document as of 2002. We still think that these are important since colleagues teaching application subjects want to be sure that students have at least an "initial familiarity" with certain mathematical concepts and procedures they need in their application modelling (as Artigue, Batanero & Kent (2007, p.1034) put it: "The right balance must be found"). The content-related learning outcomes are organized according to mathematical domain. In order to foster mathematical sense making, we also provide several overarching themes like "quantity" and "space and shape" for organizing these outcomes. This was also done in the OECD PISA document (OECD 2009).

In order to provide helpful orientation for designing one's own teaching, the fourth chapter outlines teaching and learning environments which might help students to obtain the competencies to an adequate degree. It is clear that such competencies cannot be obtained by just listening to lectures, so adequate forms of active involvement of students need to be installed. Topics like Transition issues, use of technology and integration of mathematics and engineering education are also discussed here. The short competency definition by Blomhoj and Jensen (2007) indicates that mathematical competency is strongly related to attitude towards mathematics since the “readiness” mentioned in the definition can be expected when one has a somewhat positive attitude with respect to its helpfulness. Therefore, this chapter concludes with an outline of the attitude towards mathematics that we wish engineering students to develop.

Quite understandably, students are also oriented towards getting good marks. Therefore, the assessment procedures determine to a good extent the behaviour of students and are hence important for really achieving progress in competencies. Chapter 5 outlines different forms of assessment which might be adequate for capturing certain kinds of achievements. It also discusses the role of technology in assessment and the question of identifying requirements for passing.

The current curriculum document does not prescribe a specific degree of progress relating to mathematical competences or a determined set of content-related learning outcomes. The engineering profession and hence engineering study programmes at university are far too heterogeneous to identify one profile for all. The main purpose of this document is to provide orientation for those who set up concrete mathematics curricula for their specific engineering programme. The competence framework should help to avoid an approach that is mainly restricted to contents. It can be used to analyze existing curricula and to design new ones. It helps institutions and lecturers to identify their own profile in that it facilitates the description of the role and importance of different competencies and hence their weighting within a study programme. Having this in addition to the profile concerning the content-related learning outcomes organized in chapter 3, the intention stated for the second edition of this curriculum is still valid but from a much broader perspective: “This curriculum is intended as a benchmark by which higher education institutions in Europe can judge the mathematics provision in their engineering undergraduate degree programmes.” (Mustoe & Lawson 2002, p. 2).

In recent investigations (Cardella 2008; Barker et al. 2004; Ganter & Barker 2004) the importance of having close contacts between lecturers in mathematics and engineering departments was emphasized. The competencies can also serve for discussing with engineering lecturers in which ways the mathematical education of engineers is distributed between mathematics and application subjects. The second edition of the curriculum (2002) already states – from a content-related point of view – that many of the topics listed on level 3 of the curriculum will rather be taught “as part of units on the engineering topics to which they directly apply.” (Mustoe & Lawson 2002, p. 45). This is not only true with respect to contents but definitely also with respect to mathematical competencies (or mathematical thinking, cf. Cardella 2008, p. 153). Considering e.g. the modelling competency, setting up models and solving problems within models is certainly an important activity in engineering mechanics and in many other engineering subjects that make heavy use of mathematics. Having experienced the usage of a mathematical concept in different application subjects definitely adds to the mathematical competence of a student in that it makes a concept more meaningful and also helps to develop an attitude towards mathematics where the role of the latter is perceived as potential problem solver.

Finally, this document is not meant to be a “Handbook for the mathematical education of engineers”. Nevertheless, it intends to give support for thinking about many aspects of mathematics education like learning environments and assessment since these are quite important for achieving the competencies stated in chapter 2. In this document we merely want to give an overview and to provide some guidance. Many of the issues are, and will be, discussed in journal articles and contributions to seminars of the working group. The reader is advised to consult the group's webpage for such material and current discussions (sefi.htw-aalen.de).

2 General Mathematical Competencies for Engineers

As was already stated in the introduction, we adopt the definition of mathematical competence used in the Danish KOM project. Hence, we define **mathematical competence** as "*the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role*" *(Niss 2003a, p.6/7).* In this chapter we first give an overview of the constituents of such competence, the mathematical competencies identified within the Danish KOM project, and explain their meaning. These competencies are not "binary variables" which are present or not. In order to describe the extent to which competencies do or should exist, the KOM project uses three dimensions: degree of coverage, radius of action and technical level. These are also explained in the next section. Moreover, the OECD PISA document (OECD 2009) provides a structuring of the competencies into different levels, called "clusters" which will be adapted to the mathematical education of engineers in the next section. The second section provides examples of the competencies in engineering contexts. This serves to illustrate how the competency concept can actually be used in concrete engineering education settings to describe the goals of mathematics education. The third section then gives some information on how to specify educational profiles using the competency framework.

2.1 Competencies, Dimensions, and Clusters

In order to specify in more detail what mathematical competence is, the KOM project set up a list of eight competencies which together constitute the overall competence. We reproduce a slightly modified version of this list and give some short explanations based on (Niss 2003a, p. 7-9). For a more detailed description we refer the reader to (Niss 2003a,b) and the final report of the KOM group (Niss & Jensen 2010). Moreover, the illustrative examples in the next section should provide more insight into the meaning of the competencies and their application to engineering education.

1. Thinking mathematically
   This competency comprises a knowledge of the kind of questions that are dealt with in mathematics and the types of answers mathematics can and cannot provide, and the ability to pose such questions. It includes the recognition of mathematical concepts and an understanding of their scope and limitations as well as extending the scope by abstraction and generalization of results. This also includes an understanding of the certainty mathematical considerations can provide.
2. Reasoning mathematically
   This competency includes on the one hand the ability to understand and assess an already existing mathematical argumentation (chain of logical arguments), in particular to understand the notion of proof and to recognize the central ideas in proofs. It also includes the knowledge and ability to distinguish between different kinds of mathematical statements (definition, if-then-statement, iff-statement etc.). On the other hand, it includes the construction of own chains of logical arguments and hence of transforming heuristic reasoning into own proofs (reasoning logically).
3. Posing and solving mathematical problems
   This competency comprises on the one hand the ability to identify and specify mathematical problems (be they pure or applied, open-ended or closed) and on the other hand the ability to solve mathematical problems (including knowledge of the adequate algorithms). What really constitutes a problem is not well-defined and it depends on personal capabilities

whether or not a question is considered as a problem. This has to kept in mind, for example when identifying problems for a certain group of students.

4. Modelling mathematically
   This competency also has essentially two components: The ability to analyze and work in existing models (find properties, investigate range and validity, relate to modeled reality) and the ability to "perform active modelling" (structure the part of reality that is of interest, set up a mathematical model and transform the questions of interest into mathematical questions, answer the questions mathematically, interpret the results in reality and investigate the validity of the model, monitor and control the whole modelling process). This competency has been investigated in more detail in (Blomhoj & Jensen 2003, 2007).
5. Representing mathematical entities
   This competency includes the ability to understand and use mathematical representations (be they symbolic, numeric, graphical and visual, verbal, material objects etc.) and to know their relations, advantages and limitations. It also includes the ability to choose and switch between representations based on this knowledge.
6. Handling mathematical symbols and formalism
   This competency includes the ability to understand symbolic and formal mathematical language and its relation to natural language as well as the translation between both. It also includes the rules of formal mathematical systems and the ability to use and manipulate symbolic statements and expressions according to the rules.
7. Communicating in, with, and about mathematics
   This competency includes on the one hand the ability to understand mathematical statements (oral, written or other) made by others and on the other hand the ability to express oneself mathematically in different ways.
8. Making use of aids and tools
   This competency includes knowledge about the aids and tools that are available as well as their potential and limitations. Additionally it includes the ability to use them thoughtfully and efficiently.

These competencies are overlapping (i.e. aspects of one competency are also needed within another, for example to express oneself using symbols one needs the competency of handling mathematical symbols) but emphasize different aspects and are therefore separated. They can be organized in two groups. Competencies 1 to 4 make up "the ability to ask and answer questions in and with mathematics" (Niss 2003a, p. 7) whereas competencies 5 to 8 are concerned with "the ability to deal with and *manage mathematical language and tools*" (Niss 2003a, p. 8). The list is not derived from theoretical considerations. Its value lies in leading the thinking process about what we want to achieve in the mathematical education of engineers to abilities that are widely accepted as being important. This value will become evident in the next section when we present examples.

If one wants to state for a certain mathematical competency to which extent students should have obtained it at a certain stage of their mathematical education, one needs some criteria or dimensions for specifying this. In the KOM project, three different dimensions for specifying and measuring progress are introduced (Niss 2003a, p. 10):

- Degree of coverage "is the extent to which the person masters the characteristic aspects" of a competency. In the short descriptions given above one can already recognize that a competency consists of or includes a bundle of components. For example, there often is an "analytic" side related to understanding and analyzing existing "objects" (expressions, proofs, models etc.) and a "constructive" side related to setting up one's own "objects" (chains of reasoning, models, texts etc.). The coverage then might be focus on one of these sides.

- The Radius of action comprises the "contexts and situations in which a person can activate" a competency. If, for example, the modelling competency is restricted to growth or decline situations then this should be stated using the "radius of action". If symbolic manipulation of functions is only possible when the independent variable is x and the dependent one is y, this is also a restriction of the radius of action.
- The Technical level "indicates how conceptually and technically advanced the entities and tools are with which the person can activate the competence". For example, the modelling of growth can be restricted to linear models or the usage of symbolic expressions for the computation of areas can be restricted to formulae for simple geometric figures (excluding expressions using integrals).

For the modelling competency, a more comprehensive investigation and exemplification of these dimensions can be found in (Blomhoj & Jensen 2007). Having a clear perception of the desired progress regarding the dimensions is an important prerequisite for setting up learning environments (chapter 4) and assessment regimes (chapter 5). In order to specify the desired degree of coverage, one can also use the levels described in the OECD PISA document (OECD 2009). In the following we describe these so-called clusters for the field of engineering mathematics. For specifying a mathematics curriculum for a study course in engineering, one has to determine the importance of each cluster in each of the eight competencies and then explain to which extent the levels should be achieved in an exemplary way.

1. Thinking mathematically

Reproduction: This includes the recognition of mathematical questions which were similarly posed in earlier educational settings and the ability to recall potential answers. For example, the content-related abilities specified in chapter 3 belong mainly to the reproduction cluster. This also includes the recognition of mathematical concepts and procedures in application contexts when these have been handled formerly in application subjects like engineering mechanics. The reproduction cluster also comprises a rough idea of different mathematical statements like definitions which are "free", and assertions that have to be at least roughly justified because this distinction has been emphasized in the previous educational process.

Connections: This cluster includes the recognition of mathematical questions in situations which are at least slightly different from those known from former educational settings. For example, when students have learned to use a mathematical model in certain mechanical configurations (like trusses) then they should expect that in similar contexts there should also be a mathematical model which allows to answer the mechanical questions of interest. The connections cluster also includes the posing of mathematical questions which are only similar to those encountered before.

Reflection: This cluster includes the recognition of mathematical concepts and the value of mathematisation in a new or more complex situation such that students have to reflect on similarities or analogies with other situations and this way come to the assumption that mathematics should help. This also includes finding the mathematical kernel of a question. It also includes reflections on what can and what cannot be achieved by a mathematical approach (e.g. using a complicated mathematical model for which no model parameters are available won't be particularly helpful; a rough model will be applicable but it will only provide approximative results). This also includes seeing the same mathematical question in different contexts.

2. Reasoning mathematically

Reproduction: This includes the reproduction of arguments, chains of arguments and solution procedures learnt before (for example reproduce the arguments for finding the solution type and the solution for a linear system of equations). The arguments are non-formally formulated, students are rather required to recall the main "line" of argumentation. The students are able to follow chains of arguments in well-known application contexts (e.g. mathematical reasoning in mechanical engineering leading to a formula or a mathematical model, or a way of computing interesting quantities).

Connections: This cluster includes the connection of well-known arguments to new chains or the application to different contexts (for example using well-known geometric arguments on the specification of geometric objects in order to justify why a more complex geometric configuration is determined by some given quantities and relations). This also includes thinking about logical consequences of certain results coming from own computations or computations performed by a programme (this would then mean …). This provides the ability to check whether results can be possible or reasonable.

Reflection: This cluster includes the application of mathematical reasoning to new contexts. It also includes the ability to reason about the formal correctness of an argumentation (in the sense of proof). Moreover, it includes the reflection about the logical structure of a mathematical area (what are the main definitions and structures dealt with, what are major theorems from which the important properties follow?) and relationships/analogies between such areas (where are common or similar lines of argumentation, how can such lines be reused in other contexts?).

3. Problem solving

Reproduction: This includes the ability to recognize and solve well-practised closed-form problem types (most of those which can be found in the list of learning outcomes in chapter 3) where the solution can be obtained by using well-trained procedures which were specifically learnt for solving the problem type.

Connections: This includes the connection of well-known solution procedures in order to solve problems which are at least slightly different from those practised in the educational process before and/or are more open. It also includes the ability to apply well-known solution procedures in contexts different from those where they were first learnt.

Reflection: This includes posing and solving mathematical problems which are different from those encountered before and potentially more open. It also includes reflection on the nature and kernel of the problem, on similarities and common features of problems, on available problem solving strategies and on possible solutions. It also includes the application of more general problem solving strategies to work on a problem (for example: look for special cases first and then try to combine the general case from special cases).

4. Modelling mathematically

Reproduction: This includes the ability to recognize simple, well-structured situations which have been modeled in a similar way before in the educational process. The solution of mathematical problems within the model and the translation between model and real situation have been practised before.

Connections: This includes the use of familiar mathematical models in situations which are different from those encountered before. Moreover, it includes the use of existing modelling means and principles for setting up own (not too complex) models. It also includes the interpretation and validation of solutions in not entirely familiar contexts.

Reflection: This includes a reflection on adequate modelling means and models and setting up more complex non-familiar models. It includes an interpretation of the results of the work within the mathematical model regarding the real situation that has been modeled, and a corresponding reflection of the validity of the model. It also includes a reflection about the modelling process itself and the ability to describe and justify modelling decisions. Moreover, the reflection cluster contains the ability to understand complex models set up by others and their justification of these models.

5. Communication

Reproduction: This includes the citation of basic definitions and properties and the communication of basic computations and their results (in the original context where they were learnt), both orally and in written form. It also includes the understanding of these when presented by others, particularly example computations in text books. It also includes the ability to understand well-known formulae in formularies (mathematical ones or ones for an application subject).

Connections: This includes the oral and written description of more complicated computations and mathematical argumentations. It also includes the description and understanding of mathematical work in application areas where it has not been trained before. This includes also the ability to describe and explain the set up of mathematical models and the work within these models, for example parameter variations. It also includes the understanding of mathematical work in application texts, lectures and talks.

Reflection: This includes the ability to find an adequate level of description (depth, detailedness) when presenting (orally or in written form) mathematical content (concepts, models, computations) to an audience. This also includes the ability to explain one’s own mathematical work and to understand mathematical explanations given by others. It includes the description of contexts containing more complex relationships, including logical one’s, and the understanding of such descriptions provided by others.

6. Representing mathematical entities

Reproduction: This includes the usage and understanding of familiar representations of well-known mathematical objects in well-known contexts which have been dealt with in the former mathematical education process. It also includes the switching between these in ways that have been practised before.

Connections: This includes the ability to work with less familiar representations. This also includes switching between representations which has not been entirely trained before. It includes the work with different representations in new application contexts, particularly the ability to find an adequate representation which can be well interpreted in an application context.

Reflection: This includes a reflection of advantages and disadvantages of representations and a creative combination of and switching between representations in order to take the maximum advantage. It also includes the invention of non-standard ones or variation of standard ones if this is necessary or helpful. It also includes the understanding of non-familiar representations set up by others.

7. Handling mathematical symbols and formalism

Reproduction: This includes the ability to decode and interpret basic symbolic and formal language as practiced before in well-known situations. It also includes the manipulation of symbols (performing mathematical operations) in ways formerly practised (solving a certain kind of equation, simplifying an expression, computing the symbolic solution of a linear differential equation with constant coefficients). Moreover, the use of logical symbols (like => and ⇔) should also be understood in argumentations experienced before.

Connections: This includes the recognition, decoding and interpretation of symbolic and formal language in less well-known contexts. It also includes the creation and manipulation of statements and expressions by combining familiar handling procedures.

Reflection: This includes the recognition, decoding and interpretation of symbolic and formal language in new contexts. It also includes the handling of complex statements and expression where deeper reflections are necessary to understand, transform or manipulate them. Moreover, it includes the translation between formal language and natural language.

8. Making use of aids and tools

Reproduction: This includes the ability to use familiar aids and tools (text book, formulary, pocket calculator, CAS, application programmes based on mathematical concepts which have been used in the former mathematical or application education) in ways close to those practised before.

Connections: This includes the ability to use familiar aids and tools in new contexts and situations (“connecting” a familiar tool with an unfamiliar context). For example, one is able to use a formulary to look up formulae for situations not encountered and practised before.

Reflection: This includes the usage of unfamiliar aids and tools in new and known contexts. For example, the usage of a new programme for a known context. It also includes knowledge about the potential abilities and limitations of aids and tools: What should a programme "in principle" be able to do, what is not possible in general? This also includes a reflection on which tool to use for which task as well as a reflection of interfaces between tools and ways and adequate situations to interconnect and combine them.

It is important to have a clear understanding of the relationship between mathematical contents/topics and competencies in order to recognize the role contents play in competency-based curricula. (Niss 2003a, p. 10) suggests to view competencies and mathematical topic areas as "orthogonal", i.e. to specify "how the corresponding competency manifests itself when dealing with the corresponding topic at the educational level at issue". Having the dimensions describing the extent to which a competency is present at hand, one can be a bit more specific: Content-related abilities and hence contents appear in the dimension "technical level" where the mathematical entities and operations to which the competency can be applied are to be specified. In some examples in (Niss & Jensen 2010) the radius of action also includes contents, e.g. when for the problem posing and solving competency different mathematical areas are named to indicate the radius of action.

2.2 Examples

In this section we clarify the competency concept by presenting some example tasks from engineering where the competencies are necessary for successful work. The examples show on the one hand what we want students to be able and willing to do, and on the other hand what might be adequate assignments for learning, i.e. for obtaining the competencies. Competencies are about acting "in response to a certain kind of mathematical challenge of a given situation". The situations in which students experience such challenges usually might be within a mathematical environment or occur in application subjects that are part of their study courses. Therefore, we present in the following a purely mathematical example as well as examples which are taken from engineering assignments or text books on application subjects like machine elements. We cover the main areas of application (mechanical, electrical, civil engineering). Since we consider such examples as very important for understanding the competence concept and for getting ideas to implement it in ones own teaching, we will provide a collection of additional examples on the group's webpage (http://sefi.htw-aalen.de).

1. On a slide there are the graph of a function and several candidates for the graph of the derivative. Discuss with your neighbor which one is the correct one and give your vote in a voting system.

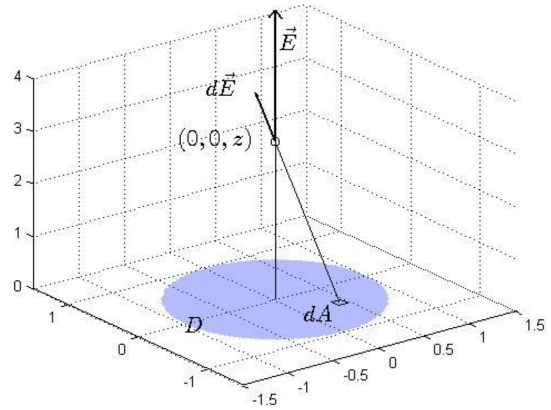
Here, students have to reason about the properties of the function and corresponding properties of the derivative and its graph (reasoning) in order to find the correct candidate. They also have to communicate their line of argumentation to their neighbor (communicating). Moreover, for solving the problem they can think about strategies like "look for simple properties which should be there but which are not (exclusion principle) in order to remove candidates from the list" (problem solving).

2. Consider two gears with tooth numbers m and n (see picture below). Each tooth in one gear should meet each tooth of the other one (and not just a subset) in order to have equally distributed abrasion and low noise excitation. How does this affect the choice of tooth numbers?

Reading this task a student should think that it has to do with integers and relationships between integers, so mathematics should provide an answer (thinking mathematically). Then, the requirement on the meeting of teeth has to be translated into a mathematical condition including m and n applying a respective chain of arguments (reasoning mathematically): Say, tooth 1 of gear one meets first tooth 1 of gear two, then tooth 1+m, 1+2m, 1+3m, …, i.e. 1+r*m modulo n. So, the condition is equivalent to "there are $r,k \in \mathbb{Z}: 1+rm = s+kn$ for s=1,…,n". This is equivalent to "there are $r,k \in \mathbb{Z}: rm = s+kn$ for s=0,…,n-1" which in turn is equivalent to "there are $r,k \in \mathbb{Z}: rm = 1+kn$" (thinking and reasoning mathematically). Having this condition, one has to solve the problem for which m and n the condition is fulfilled (here: m and n must be relatively prime, i.e. they have no common factor except 1) (posing and solving mathematical problems). Another way of tackling this task might be to get a book on machine elements, find and understand the respective information in this book (making use of aids and tools, communication in, with, about mathematics).

3. A thin circular disc has an evenly distributed charge. Find the electrostatic field at an arbitrary point above the center of the disc.

To solve this problem, the student first must understand that mathematics can do the job. First the real world problem should be transformed into a mathematical one. The real object, a thin disc is represented by a mathematical object, the set D of points subject to the conditions: $x^2+y^2 \leq R^2$, z = 0, where R is the unspecified radius of the disc. The, also unspecified, charge Q is evenly distributed which implies that the surface charge density is constant $\sigma=Q/\pi R^2$ [C/m^2] (thinking mathematically).



Then some mathematical modelling should take place; Coulomb´s law $F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2}$ implies that the electrostatic field at a point P, caused by the charge in a small area, dA, is $d\vec{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\sigma dA}{d^2}\vec{e}$, where $\hat{e}$ is a unit vector pointing from the small surface area towards the point and d is the distance between the small area and the point. The superposition principle in physics implies that the electrostatic field is then given by an integral $\vec{E} = \iint_D \frac{1}{4\pi\varepsilon_0} \cdot \frac{\sigma dA}{d^2}\vec{e}$.

By rotational symmetry the field at a point on the z-axis is directed along the axis. Thus we only have to take the vertical component of $\hat{e}$ into account and the magnitude of the field is $\left|\vec{E}\right| = \frac{\sigma}{4\pi\varepsilon_0} \cdot \iint_D \frac{z}{\sqrt{z^2 + r^2}} \cdot \frac{dA}{z^2 + r^2}$, where r is the distance from the point in the disc to the origin (reasoning mathematically, posing and solving mathematical problems).

Polar coordinates and a straightforward calculation then give the answer
$\left|\vec{E}\right| = \frac{Q}{2\pi R^2 \varepsilon_0} \cdot \left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right)$(handling mathematical symbols and formalism).

The problem can be altered in order to add more competencies. Moving the point P away from the axis or altering the distribution of the charge may lead to integrals which cannot easily be calculated by hand (making use of aids and tools).

The problem presented here can be found in any textbook in electrostatics, but altered in a suitable way as suggested above or by altering the charged surface to some surface in space (for instance a spherical shell or a torus), the problem turns into a project which can be reported in a student group. The report may well include a presentation using graphics (communicating in, with, and about mathematics, representing mathematical entities).

4. Example task from civil engineering (to be provided by ?)

2.3 Profiles

Since mathematical competencies are concerned with the ability to master the mathematical challenges of given situations, it is a reasonable starting point for specifying a competence profile to identify the contexts and situations where students of a certain study course meet mathematical challenges. These then determine the envisaged radii of action for the competencies to be listed later. One can then specify in more detail the mathematical concepts and procedures occurring in the challenges identified before as well as the corresponding abilities (technical level) and finally elaborate in more detail the aspects of the general competencies which are involved (degree of coverage, level). As an example, we roughly sketch below how such a profile could look like for the mathematical education of practice-oriented mechanical engineers aiming at a Bachelor degree at a university of applied sciences.

Concerning the mathematical challenges such students of mechanical engineering meet, it seems reasonable to inspect the application subjects occurring in the study course and search for those challenges. For taking into account later challenges showing up at the workplace, workplace studies are required but this is a field where much more research is needed (cf. Alpers 2010). In the sequel, we present some contexts and situations containing mathematical challenges which mainly occur in engineering mechanics, CAD, measurement and control, and machine elements and dynamics:

- Determination of loads (forces, torques) and the resulting stress and strains in machine elements or other mechanical configurations (the respective models can already be found in text books)
- Varying the dimensions of machine elements or other mechanical configurations in order to improve or even optimize certain properties (stress, weight, costs, …)
- Analysis of motion and design of motion of machines or machine parts
- Analysis of vibrations
- Modelling of controlled devices and design of controllers
- Processing of measurement data, computation of descriptive quantities and error analysis, model fitting for measured data.
- …

A systematic investigation is required for achieving a good coverage. Such an investigation can also be of great value later on when trying to find good example tasks or themes for mathematical application projects. Note that we do not assume that all mathematical challenges occurring in application subjects are handled in the mathematical part of engineering education. Nevertheless, for providing an integrated study course it is very advantageous to have a clear view of the split of responsibility.

The mathematical concepts and algorithms occurring in the identified contexts include:

- Functions and functional dependencies, construction of functions with desired properties
- Using functions for modeling behaviour (growth/decay, vibrations, logistic behaviour, …)
- Systems of equations, solution types and algorithms
- Iterative improvement and optimization algorithms
- Geometric descriptions using classical and free-form geometries and their computation
- Differentiation and integration
- Differential equations, solution types and algorithms
- Laplace transforms and working in the complex variable domain
- Fourier analysis
- Stochastic concepts like distribution, mean, variance, confidence intervals, …

- …

In the next step on the technical level one could specify in more detail the abilities choosing from those described in chapter 3.

Finally, one has to specify to which degree the eight competencies have to be covered for successfully handling the challenges. This is certainly a "non-trivial" task needing in-depth reasoning. As a first approach it is useful to specify in a coarse way the importance of each competency level (or cluster) as has been done in the table below. Note that such an importance specification should provide information on what one wants to achieve for all students. It should not prevent institutions from offering particularly talented students additional learning experiences for acquiring higher levels.

Competency \ Level	Reproduction	Connections	Reflection
Thinking math.	+	+	O
Reasoning math.	+	O	-
Problem solving	+	+	O
Modelling math.	+	+	O
Communication	+	+	O
Representation	+	+	O
Symbols and formalism	+	O	-
Aids and tools	+	+	+

Meaning of signs: +: very important, O: medium important, -: less important

Such a rough specification just gives an overview on where the emphasis lies. It needs more explanations and examples to make it suitable for designing adequate learning scenarios. In the sequel, we provide some additional explanatory material to illustrate this:

- Regarding the competency of thinking mathematically, the recognition of well-trained mathematical situations in mathematics and application subjects is very important (e.g. in machine element computations the mathematical algorithm should be recognized and a respective result should be expected). Students should also recognize the usefulness of mathematical concepts in situations similar to those encountered before (e.g. in the computation of stress for a different machine element). The recognition of the potential of mathematical work in totally new situations (e.g. recognition that a problem in a totally new context could be formulated as a mathematical optimization problem) is of medium importance.
- Regarding the competency of reasoning mathematically, students should be able use arguments they have trained before in very similar situations (e.g. argue how the solution process of a linear system of equation works and why one ends up in a certain solution situation). They should also understand a well-known, not too-complicated mathematical argumentation in an application text book. The connections level is of medium importance. Students should e.g. connect simple geometric arguments for determining whether a geometric configuration is fully specified by giving some data or deduce consequences from programme results in order to perform plausibility checks. The reflection level is of less importance since the engineering students in this profile are not required to set up chains of advanced mathematical reasoning (e.g. to write an article in theoretical mechanics).

- Regarding the competency of problem solving, students should be able to recognize the problems and perform the problem solving procedures learnt before (see e.g. chapter 3). They should also be able to recognize and solve similar problems in different contexts (e.g. solve a linear DE with constant coefficients in a different context to the one encountered before). They should also be able to work on more open design-type mathematical questions, e.g. the design of a motion function with certain properties or the design of a machine element fulfilling certain restrictions regarding stress and geometry. Students should also have – to a moderate extent – reflective capabilities concerning problem solving strategies, e.g. see strategies for parameter variation in order to improve output values. They are not required to work on harder mathematical problems for which new strategies are necessary.
- Regarding the competency of mathematical modelling, students should be able to recognize and solve problems in standard engineering models developed mostly in application subjects like mechanics or control theory. They should be able to interpret the results as trained before. Moreover, they should use well-known modelling means (like forces, torques, equilibrium principle, cutting principle) learnt before (mostly in application subjects) to set up models in different situations (e.g. model a mechanical configuration by identifying important forces and using equilibrium equations; use the function concept for modelling motion). Students are not expected to develop new modelling means but they should be able to reflect on useful simplifications and on the validity of a simplified model.
- Regarding the competency of communication, students should be able to use simple, mostly informal mathematical language encountered before in their mathematical education and they should be able to understand such language in text books, particularly in application text books and formularies. They should be able to informally describe orally and in written form their argumentation or procedure for solving a mathematical problem (like parameter variation to improve a property) or for setting up and working within a mathematical model. It is less important to communicate using formal mathematical language or to communicate more complex logical argumentations.
- Regarding the competency of representing mathematical entities, students should be able to understand and use standard representations which they learnt before and encountered or will encounter in application text books (different representations of functions, of geometric entities, but also more advanced representations of signal functions in the frequency domain). They should be able to extract information from such representations and switch to a particularly meaningful one. They also should be able to understand well-known forms of representation in new contexts. Students should be able to reflect about advantages and disadvantages of representations but they do not have to invent new ones.
- Regarding the competency of handling mathematical symbols and formalism, students should be able to reproduce the symbols and formalism handling encountered before (see chapter 3), i.e. recognize the symbols and be able to perform operations practised before. They should be able to do this in contexts that are not totally familiar and where different notations are used (in the simplest way s(t) for motion instead of y(x)). They are not required to have a deeper understanding of logical formalism (using implication and equivalent symbols is sufficient). There also is no necessity of being able to perform large and complex formal computations by hand. Invention of new symbols and formalism (as usually done by mathematicians) is also of no importance.

- Regarding the competency of using aids and tools, this is of particular importance in engineering work environments where computer programmes are ubiquitous. Students should be able to use mathematics programmes to solve the mathematical problems dealt with in the mathematical education, and they should be able to specify mathematical input and interpret mathematical output of application programmes, as learnt before. They should also be able to use programmes in new contexts to solve problems similar to those handled before (e.g. solve a new differential equation with a mathematics programme). They should also be able to use books or other text sources (e.g. internet pages) to look up computations or mathematical results that are helpful for working on their application problems (e.g. look up the relationship between tooth numbers in example 2 in the previous section). They should also learn to reflect on what one can expect from a programme based on a certain mathematical model for the application situation. Moreover, they should be able to check the correct working of a programme by using simple examples computed by hand.

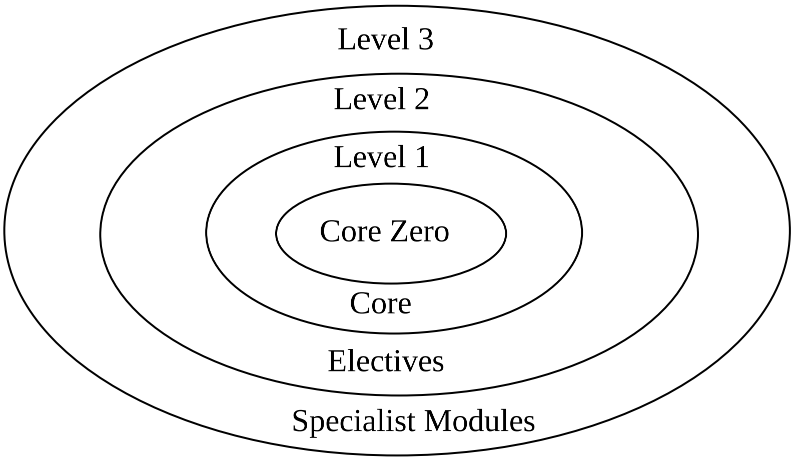
The above sketch is meant to provide a rough impression of how a concrete curriculum based on this document can look like. Considerably more effort is necessary to complete this to a full specification.

3 Content-related competencies, knowledge, and skills

In the first OECD PISA study and in national or regional curricula developed in response to the results of the study (e.g. (KMK 2003), (Ministry of Education Baden-Württemberg 2004)), more detailed content-related competencies, knowledge and skills have been identified since the general competencies are not sufficient for guiding teaching in the secondary education environment. In this chapter we apply a similar approach for engineering education. The next section 3.1 states content-related learning outcomes which are organized in different levels and according to mathematical domain. In order to foster mathematical sense-making, we also provide in section 3.2 several overarching themes like “quantity” and “space and shape” for organizing these outcomes.

3.1 Learning outcomes regarding knowledge and skills

In this section content-related learning outcomes are presented. It has been arranged in a structure which has four levels. These levels represent an attempt to reflect the hierarchical structure of mathematics and the way in which mathematics can be linked to real applications of ever-greater sophistication as the student progresses through the engineering degree programme.



Schematic diagram of the proposed structure

A schematic diagram of the proposed structure is shown in the above figure. Note that there is a central core of material essential for all engineering graduates. Typically this core material would be covered by the end of the first year but the teaching of some of it might extend into the second year of a course. Realistic pre-requisite knowledge (Core Zero) would be assumed. Core Zero, as set out in Section 3.2, does not represent the minimum which can be assumed in every European country. Instead it is covers those topics which make up an essential foundation for Core Level 1 and beyond. It is likely that most institutions will need to teach explicitly some of Core Zero topics whilst other institutions may have a parallel programme of support classes or clinics to help students who may be deficient in some areas.

Core Zero is specified in Section 3.1.1 in considerable detail. We make no apology for this: it comprises such essential material that no engineering student can afford to be deficient in these topics. The knowledge and skills in mathematics of a student entering tertiary education is not easily predicted from the qualifications achieved prior to entry and some kind of diagnostic testing and additional support may be needed. This support may well be needed throughout the first year of an engineering degree programme.

Core level 1 comprises the knowledge and skills which are necessary in order to underpin the general Engineering Science that is assumed to be essential for most engineering graduates. Items of basic knowledge will be linked together and simple illustrative examples will be used. It should be pointed out that the mathematical needs of Computer Science and Software Engineering are markedly different from all other branches of engineering. This core curriculum is only of limited use for such courses.

Level 2 comprises specialist or advanced knowledge and skills which are considered essential for individual engineering disciplines. Synoptic elements will link together items of knowledge and the use of simple illustrative examples from real-life engineering.

Level 3 comprises highly specialist knowledge and skills which are associated with advanced levels of study and incorporates synoptic mathematical theory and its integration with real-life engineering examples. Students would progress from the core in mathematics by studying more subject-specific compulsory modules (electives). These would normally build upon the core modules and be expected to correspond to the outcomes associated with level 2 material. Such electives may build additionally on level 1, requiring knowledge of more advanced skills, and may link level 1 skills or introduce additional more engineering-specific related topics. An example of the first mode is that Mechanical Engineering students may need to study the vibration of mechanical systems through the applied use of systems of ordinary differential equations. Here, topics that build on foundations of differential calculus, complex numbers and matrix analysis might be expected to be covered in level 1, as these are topics probably learnt in isolation and without reference to specific engineering application. Alternatively an Electronic Engineering student may be required to learn elements of discrete mathematics directly relevant to the design and study of computer systems; these would probably be unsuitable for core material for all engineers. This is not to say that discrete mathematics should not be taught at level 1, but the context and outcomes need to be clearly discernible within the level. Typically level 2 modules would be distributed within the second or third year of an Engineering course due to the logistics of level 1 pre-requisites.

Students within the more numerate Engineering disciplines might be expected to take further more specialised modules incorporating mathematics on an optional basis, aimed to help match their career aspirations with appropriate theoretical formation. These modules will be at an advanced level, making use of appropriate technology, and heavily influenced with examples from engineering. Teaching of these level 3 modules would be most appropriate in year 3 or 4 of a degree course. It is likely that many of these topics already exist within specialist engineering courses and typically the mathematics is embedded and taught by engineers, mathematicians or both. For some programmes meeting the highest requirements, students might be expected to study some topics close to current areas of research where the available techniques and tools may well be mathematically based.

Within the three main levels the material has been arranged under five sub-headings: analysis and calculus, discrete mathematics, geometry, linear algebra, statistics and probability. There is no intention to prescribe how the topics in the Core Curriculum should be ordered: what is offered here is a convenient grouping of items. In many cases a topic could have been placed under one of the other sub-headings. The curriculum is specified in terms of content and learning objectives. This makes the document longer, but it makes more explicit exactly what is required and is therefore more transparent for both teacher and learner.

3.1.1 Core Zero

The material in this section is the material which ideally should have been studied before entry to an undergraduate engineering degree programme. However, it is recognised that whilst there is some commonality across Europe over what is studied in pre-university mathematics, there are also a number of areas of difference. Core Zero does not consist of just those topics which are taught in school in all European countries, rather it contains material which together forms a solid platform on which to build a study of engineering mathematics at university. A consequence of this is that in many countries it will be necessary to cover some Core Zero material during the first year of a university engineering course.

The material in Core Zero has been grouped into five areas: Algebra, Analysis & Calculus, Discrete Mathematics, Geometry & Trigonometry and Statistics & Probability. These relate to the five areas in each of the three main levels of the curriculum: Analysis & Calculus, Discrete Mathematics, Geometry, Linear Algebra and Statistics & Probability.

3.1.1.1 Algebra

Arithmetic of real numbers

As a result of learning this material you should be able to

- carry out the operations add, subtract, multiply and divide on both positive and negative numbers
- express an integer as a product of prime factors
- calculate the highest common factor and lowest common multiple of a set of integers
- obtain the modulus of a number
- understand the rules governing the existence of powers of a number
- combine powers of a number
- evaluate negative powers of a number
- express a fraction in its lowest form
- carry out arithmetic operations on fractions
- represent roots as fractional powers
- express a fraction in decimal form and vice-versa
- carry out arithmetic operations on numbers in decimal form
- round numerical values to a specified number of decimal places or significant figures
- understand the concept of ratio and solve problems requiring the use of ratios
- understand the scientific notation form of a number
- manipulate logarithms
- understand how to estimate errors in measurements and how to combine them.

Algebraic expressions and formulae

As a result of learning this material you should be able to

- add and subtract algebraic expressions and simplify the result
- multiply two algebraic expressions, removing brackets
- evaluate algebraic expressions using the rules of precedence
- change the subject of a formula
- distinguish between an identity and an equation
- obtain the solution of a linear equation
- recognise the kinds of solution for two simultaneous equations

- understand the terms direct proportion, inverse proportion and joint proportion
- solve simple problems involving proportion
- factorise a quadratic expression
- carry out the operations add, subtract, multiply and divide on algebraic fractions
- interpret simple inequalities in terms of intervals on the real line
- solve simple inequalities, both geometrically and algebraically
- interpret inequalities which involve the absolute value of a quantity.

Linear laws

As a result of learning this material you should be able to

- understand the Cartesian co-ordinate system
- plot points on a graph using Cartesian co-ordinates
- understand the terms ‘gradient’ and ‘intercept’ with reference to straight lines
- obtain and use the equation $y = mx + c$
- obtain and use the equation of a line with known gradient through a given point
- obtain and use the equation of a line through two given points
- use the intercept form of the equation of a straight line
- use the general equation $ax + by + c = 0$
- determine algebraically whether two points lie on the same side of a straight line
- recognise when two lines are parallel
- recognise when two lines are perpendicular
- obtain the solution of two simultaneous equations in two unknowns using graphical and algebraic methods
- interpret simultaneous linear inequalities in terms of regions in the plane
- reduce a relationship to linear form.

Quadratics, cubics, polynomials

As a result of learning this material you should be able to

- recognise the graphs of $y = x^2$ and $y = -x^2$
- understand the effects of translation and scaling on the graph of $y = x^2$
- rewrite a quadratic expression by completing the square
- use the rewritten form to sketch the graph of the general expression $ax^2 + bx + c$
- determine the intercepts on the axes of the graph of $y = ax^2 + bx + c$
- determine the highest or lowest point on the graph of $y = ax^2 + bx + c$
- sketch the graph of a quadratic expression
- state the criterion that determines the number of roots of a quadratic equation
- solve the equation $ax^2 + bx + c = 0$ via factorisation, by completing the square and by the formula
- recognise the graphs of $y = x^3$ and $y = -x^3$
- recognise the main features of the graph of $y = ax^3 + bx^2 + cx + d$
- recognise the main features of the graphs of quartic polynomials
- state and use the remainder theorem
- derive the factor theorem

- factorise simple polynomials as a product of linear and quadratic factors.

3.1.1.2 Analysis and Calculus

Functions and their inverses

As a result of learning this material you should be able to

- define a function, its domain and its range
- use the notation f(x)
- determine the domain and range of simple functions
- relate a pictorial representation of a function to its graph and to its algebraic definition
- determine whether a function is injective, surjective, bijective
- understand how a graphical translation can alter a functional description
- understand how a reflection in either axis can alter a functional description
- understand how a scaling transformation can alter a functional description
- determine the domain and range of simple composite functions
- use appropriate software to plot the graph of a function
- obtain the inverse of a function by a pictorial representation, graphically or algebraically
- determine the domain and range of the inverse of a function
- determine any restrictions on f(x) for the inverse to be a function
- obtain the inverse of a composite function
- recognise the properties of the function 1/x
- understand the concept of the limit of a function.

Sequences, series, binomial expansions

As a result of learning this material you should be able to

- define a sequence and a series and distinguish between them
- recognise an arithmetic progression and its component parts
- find the general term of an arithmetic progression
- find the sum of an arithmetic series
- recognise a geometric progression and its component parts
- find the general term of a geometric progression
- find the sum of a finite geometric series
- interpret the term 'sum' in relation to an infinite geometric series
- find the sum of an infinite geometric series when it exists
- find the arithmetic mean of two numbers
- find the geometric mean of two numbers
- obtain the binomial expansions of $(a+b)^s$, $(1+ x)^s$ for s a rational number
- use the binomial expansion to obtain approximations to simple rational functions.

Logarithmic and exponential functions

As a result of learning this material you should be able to

- recognise the graphs of the power law function
- define the exponential function and sketch its graph
- define the logarithmic function as the inverse of the exponential function
- use the laws of logarithms to simplify expressions
- solve equations involving exponential and logarithmic functions
- solve problems using growth and decay models.

Rates of change and differentiation

As a result of learning this material you should be able to

- define average and instantaneous rates of change of a function
- understand how the derivative of a function at a point is defined
- recognise the derivative of a function as the instantaneous rate of change
- interpret the derivative as the gradient at a point on a graph
- distinguish between 'derivative' and 'derived function'
- use the notations dy/dx , f ′(x) , y′ etc.
- use a table of the derived functions of simple functions
- recall the derived function of each of the standard functions
- use the multiple, sum, product and quotient rules
- use the chain rule
- relate the derivative of a function to the gradient of a tangent to its graph
- obtain the equation of the tangent and normal to the graph of a function.

Stationary points, maximum and minimum values

As a result of learning this material you should be able to

- use the derived function to find where a function is increasing or decreasing
- define a stationary point of a function
- distinguish between a turning point and a stationary point
- locate a turning point using the first derivative of a function
- classify turning points using first derivatives
- obtain the second derived function of simple functions
- classify stationary points using second derivatives.

Indefinite integration

As a result of learning this material you should be able to

- reverse the process of differentiation to obtain an indefinite integral for simple functions
- understand the role of the arbitrary constant
- use a table of indefinite integrals of simple functions
- understand and use the notation for indefinite integrals
- use the constant multiple rule and the sum rule

- use indefinite integration to solve practical problems such as obtaining velocity from a formula for acceleration or displacement from a formula for velocity.

Definite integration, applications to areas and volumes

As a result of learning this material you should be able to

- understand the idea of a definite integral as the limit of a sum
- realise the importance of the Fundamental Theorem of the Calculus
- obtain definite integrals of simple functions
- use the main properties of definite integrals
- calculate the area under a graph and recognise the meaning of a negative value
- calculate the area between two curves
- calculate the volume of a solid of revolution
- use trapezium and Simpson’s rules to approximate the value of a definite integral.

Complex numbers

As a result of learning this material you should be able to

- define a complex number and identify its component parts
- represent a complex number on an Argand diagram
- carry out the operations of addition and subtraction
- write down the conjugate of a complex number and represent it graphically
- identify the modulus and argument of a complex number
- carry out the operations of multiplication and division in both Cartesian and polar form
- solve equations of the form $z^n = a$, where a is a real number.

Proof

As a result of learning this material you should be able to

- distinguish between an axiom and a theorem
- understand how a theorem is deduced from a set of axioms
- appreciate how a corollary is developed from a theorem
- follow a proof of Pythagoras’ theorem
- follow proofs of theorems for example, the concurrency of lines related to triangles and/or the equality of angles related to circles.

3.1.1.3 Discrete Mathematics

Sets

As a result of learning this material you should be able to

- understand the concepts of a set, a subset and the empty set
- determine whether an item belongs to a given set or not
- use and interpret Venn diagrams

- find the union and intersection of two given sets
- apply the laws of set algebra.

3.1.1.4 Geometry and Trigonometry

Geometry

As a result of learning this material you should be able to

- recognise the different types of angle
- identify the equal angles produced by a transversal cutting parallel lines
- identify the different types of triangle
- state and use the formula for the sum of the interior angles of a polygon
- calculate the area of a triangle
- use the rules for identifying congruent triangles
- know when two triangles are similar
- state and use Pythagoras' theorem
- understand radian measure
- convert from degrees to radians and vice-versa
- state and use the formulae for the circumference of a circle and the area of a disc
- calculate the length of a circular arc
- calculate the areas of a sector and of a segment of a circle
- quote formulae for the area of simple plane figures
- quote formulae for the volume of elementary solids: a cylinder, a pyramid, a tetrahedron, a cone and a sphere
- quote formulae for the surface area of elementary solids: a cylinder, a cone and a sphere
- sketch simple orthographic views of elementary solids
- understand the basic concept of a geometric transformation in the plane
- recognise examples of a metric transformation (isometry) and affine transformation (similitude)
- obtain the image of a plane figure in a defined geometric transformation: a translation in a given direction, a rotation about a given centre, a symmetry with respect to the centre or to the axis, scaling to a centre by a given ratio.

Trigonometry

As a result of learning this material you should be able to

- define the sine, cosine and tangent of an acute angle
- define the reciprocal ratios cosecant, secant and cotangent
- state and use the fundamental identities arising from Pythagoras’ theorem
- relate the trigonometric ratios of an angle to those of its complement
- relate the trigonometric ratios of an angle to those of its supplement
- state in which quadrants each trigonometric ratio is positive (the CAST rule)
- state and apply the sine rule
- state and apply the cosine rule
- calculate the area of a triangle from the lengths of two sides and the included angle
- solve a triangle given sufficient information about its sides and angles

- recognise when there is no triangle possible and when two triangles can be found.

Co-ordinate geometry

As a result of learning this material you should be able to

- calculate the distance between two points
- find the position of a point which divides a line segment in a given ratio
- find the angle between two straight lines
- calculate the distance of a given point from a given line
- calculate the area of a triangle knowing the co-ordinates of its vertices
- give simple examples of a locus
- recognise and interpret the equation of a circle in standard form and state its radius and centre
- convert the general equation of a circle to standard form
- recognise the parametric equations of a circle
- derive the main properties of a circle, including the equation of the tangent at a point
- define a parabola as a locus
- recognise and interpret the equation of a parabola in standard form and state its vertex, focus, axis, parameter and directrix
- recognise the parametric equation of a parabola
- derive the main properties of a parabola, including the equation of the tangent at a point
- understand the concept of parametric representation of a curve
- use polar co-ordinates and convert to and from Cartesian co-ordinates.

Trigonometric functions and applications

As a result of learning this material you should be able to

- define the term periodic function
- sketch the graphs of sin(x), cos(x) and tan(x) and describe their main features
- deduce the graphs of the reciprocal functions cosec, sec and cot
- deduce the nature of the graphs of a*sin x , a*cos x , a*tan x
- deduce the nature of the graphs of sin(ax) , cos(ax) , tan(ax)
- deduce the nature of the graphs of sin(x + a), a + sin(x), etc
- solve the equations sin(x) = c, cos(x) = c, tan(x) = c
- use the expression a*sin(ωt +ϕ) to represent an oscillation and relate the
- parameters to the motion
- rewrite the expression a*cosωt + b*sinωt as a single cosine or sine formula.

Trigonometric identities

As a result of learning this material you should be able to

- obtain and use the compound angle formulae and double angle formulae
- obtain and use the product formulae

- solve simple problems using these identities.

3.1.1.5 Statistics and Probability

Data Handling

As a result of learning this material you should be able to

- interpret data presented in the form of line diagrams, bar charts, pie charts
- interpret data presented in the form of stem and leaf diagrams, box plots, histograms
- construct line diagrams, bar charts, pie charts, stem and leaf diagrams, box plots, histograms for suitable data sets
- calculate the mode, median and mean for a set of data items.

Probability

As a result of learning this material you should be able to

- define the terms 'outcome', 'event' and 'probability'.
- calculate the probability of an event by counting outcomes
- calculate the probability of the complement of an event
- calculate the probability of the union of two mutually-exclusive events
- calculate the probability of the union of two events
- calculate the probability of the intersection of two independent events.

3.1.2 Core level 1

The material at this level builds on Core Zero and is regarded as basic to all engineering disciplines in that it provides the fundamental understanding of many mathematical principles. However, it is recognised that the emphasis given to certain topics within Core level 1 may differ according to the engineering discipline. So, for example, electrical and electronic engineers may cover some of the topics in Discrete Mathematics in greater depth than, say, Mechanical Engineers.

The material in Core level 1 can be used by engineers in the understanding and the development of theory and in the sensible selection of tools for analysis of engineering problems. This material will be taught in the early stages of a university programme. Noting the comment made in Section 3.1.1 it is possible that some of this material will be taught alongside or immediately after coverage of missing topics from Core Zero.

3.1.2.1 Analysis and Calculus

The material in this section covers the basic development of analysis and calculus consequent on the material in Core Zero.

Hyperbolic functions

As a result of learning this material you should be able to

- define and sketch the functions sinh, cosh, tanh
- sketch the reciprocal functions cosech, sech and coth
- state the domain and range of the inverse hyperbolic functions
- recognise and use basic hyperbolic identities

- apply the functions to a practical problem (for example, a suspended cable)
- understand how the functions are used in simplifying certain standard integrals.

Rational functions

As a result of learning this material you should be able to

- sketch the graph of a rational function where the numerator is a linear expression and the denominator is either a linear expression or the product of two linear expressions
- obtain the partial fractions of a rational function, including cases where the denominator has a repeated linear factor or an irreducible quadratic factor.

Complex numbers

As a result of learning this material you should be able to

- state and use Euler's formula
- state and understand De Moivre's theorem for a rational index
- find the roots of a complex number
- link trigonometric and hyperbolic functions
- describe regions in the plane by restricting the modulus and / or the argument of a complex number.

Functions

As a result of learning this material you should be able to

- define and recognise an odd function and an even function
- understand the properties 'concave' and 'convex'
- identify, from its graph where a function is concave and where it is convex
- define and locate points of inflection on the graph of a function
- evaluate a function of two or more variable at a given point
- relate the main features, including stationary points, of a function of 2 variables to its 3D plot and to a contour map
- obtain the first partial derivatives of simple functions of several variables
- use appropriate software to produce 3D plots and/or contour maps.

Differentiation

As a result of learning this material you should be able to

- understand the concepts of continuity and smoothness
- differentiate inverse functions
- differentiate functions defined implicitly
- differentiate functions defined parametrically
- locate any points of inflection of a function

- find greatest and least values of physical quantities.

Sequences and series

As a result of learning this material you should be able to

- understand convergence and divergence of a sequence
- know what is meant by a partial sum
- understand the concept of a power series
- apply simple tests for convergence of a series
- find the tangent and quadratic approximations to a function
- understand the idea of radius of convergence of a power series
- recognise Maclaurin series for standard functions
- understand how Maclaurin series generalise to Taylor series
- use Taylor series to obtain approximate percentage changes in a function.

Methods of integration

As a result of learning this material you should be able to

- obtain definite and indefinite integrals of rational functions in partial fraction form
- apply the method of integration by parts to indefinite and definite integrals
- use the method of substitution on indefinite and definite integrals
- solve practical problems which require the evaluation of an integral
- recognise simple examples of improper integrals
- use the formula for the maximum error in a trapezoidal rule estimate
- use the formula for the maximum error in a Simpson's rule estimate.

Applications of integration

As a result of learning this material you should be able to

- find the length of part of a plane curve
- find the curved surface area of a solid of revolution
- obtain the mean value and root-mean-square (RMS) value of a function in a closed interval
- find the first and second moments of a plane area about an axis
- find the centroid of a plane area and of a solid of revolution.

Solution of non-linear equations

As a result of learning this material you should be able to

- use intersecting graphs to help locate approximately the roots of non-linear equations
- use Descartes' rules of signs for polynomial equations
- understand the distinction between point estimation and interval reduction methods
- use a point estimation method and an interval reduction method to solve a practical problem

- understand the various convergence criteria
- use appropriate software to solve non-linear equations.

3.1.2.2 Discrete Mathematics

The material in this section covers the basic development of discrete mathematics consequent on the material in Core Zero.

Mathematical logic

As a result of learning this material you should be able to

- recognise a proposition
- negate a proposition
- form a compound proposition using the connectives AND, OR, IMPLICATION
- construct a truth table for a compound proposition
- construct a truth table for an implication
- verify the equivalence of two statements using a truth table
- identify a contradiction and a tautology
- construct the converse of a statement
- obtain the contrapositive form of an implication
- understand the universal quantifier “for all”
- understand the existential quantifier “there exists”
- negate propositions with quantifiers
- follow simple examples of direct and indirect proof
- follow a simple example of a proof by contradiction.

Sets

As a result of learning this material you should be able to

- understand the notion of an ordered pair
- find the Cartesian product of two sets
- define a characteristic function of a subset of a given universe
- compare the algebra of switching circuits to that of set algebra
- analyse simple logic circuits comprising AND, OR, NAND, NOR and EXCLUSIVE OR gates
- understand the concept of a countable set.

Mathematical induction and recursion

As a result of learning this material you should be able to

- understand (weak) mathematical induction
- follow a simple proof which uses mathematical induction
- define a set by induction
- use structural induction to prove some simple properties of a set which is given by induction.
- understand the concept of recursion

- define the factorial of a positive integer by recursion (any other suitable example will serve just as well).

Graphs

As a result of learning this material you should be able to

- recognise a graph (directed and/or undirected) in a real situation
- understand the notions of a path and a cycle
- understand the notion of a tree and a binary tree
- understand the notion of a binary tree.

3.1.2.3 Geometry

The material in this section covers the basic development of geometry consequent on the material in Core Zero.

Conic sections

As a result of learning this material you should be able to

- recognise the equation of an ellipse in standard form and state its foci, semiaxes and directrices
- recognise the parametric equations of an ellipse
- derive the main properties of an ellipse, including the equation of the tangent at a point
- recognise the equation of a hyperbola in standard form and find its foci, semiaxes and asymptotes
- recognise parametric equations of a hyperbola
- derive the main properties of a hyperbola, including the equation of the tangent at a point
- recognise the equation of a conic section in the general form and classify the type of conic section

3D co-ordinate geometry

As a result of learning this material you should be able to

- recognise and use the standard equation of a straight line in 3D
- recognise and use the standard equation of a plane
- find the angle between two straight lines
- find where two straight lines intersect
- find the angle between two planes
- find the intersection line of two planes
- find the intersection of a line and a plane
- find the angle between a line and a plane
- calculate the distance between two points, a point and a line, a point and a plane
- calculate the distance between two lines, a line and a plane, two planes
- recognise and use the standard equation of a singular quadratic surface (cylindrical, conical)

- recognise and use the standard equation of a regular quadratic surface (ellipsoid, paraboloid, hyperboloid).

3.1.2.4 Linear Algebra

The material in this section covers the basic development of linear algebra consequent on the material in Core Zero.

Vector arithmetic

As a result of learning this material you should be able to

- distinguish between vector and scalar quantities
- understand and use vector notation
- represent a vector pictorially
- carry out scalar multiplication of a vector and represent it pictorially
- determine the unit vector in a specified direction
- represent a vector in component form (two and three components only).

Vector algebra and applications

As a result of learning this material you should be able to

- solve simple problems in geometry using vectors
- solve simple problems using the component form (for example, in mechanics)
- define the scalar product of two vectors and use it in simple applications
- understand the geometric interpretation of the scalar product
- define the vector product of two vectors and use it in simple applications
- understand the geometric interpretation of the vector product
- define the scalar triple product of three vectors and use it in simple applications
- understand the geometric interpretation of the scalar triple product.

Matrices and determinants

As a result of learning this material you should be able to

- understand what is meant by a matrix
- recall the basic terms associated with matrices (for example, diagonal, trace, square, triangular, identity)
- obtain the transpose of a matrix
- determine any scalar multiple of a matrix
- recognise when two matrices can be added and find, where possible, their sum
- recognise when two matrices can be multiplied and find, where possible, their product
- calculate the determinant of 2 x 2 and 3 x 3 matrices
- understand the geometric interpretation of 2 x 2 and 3 x 3 determinants
- use the elementary properties of determinants in their evaluation
- state the criterion for a square matrix to have an inverse
- write down the inverse of a 2 x 2 matrix when it exists

- determine the inverse of a matrix, when it exists, using row operations
- calculate the rank of a matrix
- use appropriate software to determine inverse matrices.

Solution of simultaneous linear equations

As a result of learning this material you should be able to

- represent a system of linear equations in matrix form
- understand how the general solution of an inhomogeneous linear system of m equations in n unknowns is obtained from the solution of the homogeneous system and a particular solution
- recognise the different possibilities for the solution of a system of linear equations
- give a geometrical interpretation of the solution of a system of linear equations
- understand how and why the rank of the coefficient matrix and the augmented matrix of a linear system can be used to analyse its solution
- use the inverse matrix to find the solution of 3 simultaneous linear equations
- understand the term 'ill-conditioned'
- apply the Gauss elimination method and recognise when it fails
- understand the Gauss-Jordan variation
- use appropriate software to solve simultaneous linear equations.

Least squares curve fitting

As a result of learning this material you should be able to

- define the least squares criterion for fitting a straight line to a set of data points
- understand how and why the criterion is satisfied by the solution of $A^TAx = A^Tb$
- understand the effect of outliers
- modify the method to deal with polynomial models
- use appropriate software to fit a straight line to a set of data points.

Linear spaces and transformations

As a result of learning this material you should be able to

- define a linear space
- define and recognise linear independence
- define and obtain a basis for a linear space
- define a subspace of a linear space and find a basis for it
- define scalar product in a linear space
- understand the concept of measure
- define the Euclidean norm
- define a linear transformation between two spaces; define the image space and the null space for the transformation
- derive the matrix representation of a linear transformations

- understand how to carry out a change of basis
- define an orthogonal transformation
- apply the above matrices of linear transformations in the Euclidean plane and Euclidean space
- recognise matrices of Euclidean and affine transformations: identity, translation, symmetry, rotation and scaling.

3.1.2.5 Statistics and Probability

The material in this section covers the basic development of statistics and probability consequent on the material in Core Zero.

Data Handling

As a result of learning this material you should be able to

- calculate the range, inter-quartile range, variance and standard deviation for a set of data items
- distinguish between a population and a sample
- know the difference between the characteristic values (moments) of a population and a sample
- construct a suitable frequency distribution from a data set
- calculate relative frequencies
- calculate measures of average and dispersion for a grouped set of data
- understand the effect of grouping on these measures.

Combinatorics

As a result of learning this material you should be able to

- evaluate the number of ways of arranging unlike objects in a line
- evaluate the number of ways of arranging objects in a line, where some are alike
- evaluate the number of ways of arranging unlike objects in a ring
- evaluate the number of ways of permuting r objects from n unlike objects
- evaluate the number of combinations of r objects from n unlike objects
- use the multiplication principle for combinations.

Simple probability

As a result of learning this material you should be able to

- interpret probability as a degree of belief
- understand the distinction between a priori and a posteriori probabilities
- use a tree diagram to calculate probabilities
- know what conditional probability is and be able to use it (Bayes' theorem)
- calculate probabilities for series and parallel connections.

Probability models

As a result of learning this material you should be able to

- define a random variable and a discrete probability distribution
- state the criteria for a binomial model and define its parameters
- calculate probabilities for a binomial model
- state the criteria for a Poisson model and define its parameters
- calculate probabilities for a Poisson model
- state the expected value and variance for each of these models
- understand when a random variable is continuous
- explain the way in which probability calculations are carried out in the continuous case.

Normal distribution

As a result of learning this material you should be able to

- handle probability statements involving continuous random variables
- convert a problem involving a normal variable to the area under part of its density curve
- relate the general normal distribution to the standardised normal distribution
- use tables for the standardised normal variable
- solve problems involving a normal variable using tables.

Sampling

As a result of learning this material you should be able to

- define a random sample
- know what a sampling distribution is
- understand the term 'mean squared error' of an estimate
- understand the term 'unbiasedness' of an estimate

Statistical inference

As a result of learning this material you should be able to

- apply confidence intervals to sample estimates
- follow the main steps in a test of hypothesis.
- understand the difference between a test of hypothesis and a significance test (pvalue)
- define the level of a test (error of the first kind)
- define the power of a test (error of the second kind)
- state the link between the distribution of a normal variable and that of means of samples
- place confidence intervals around the sample estimate of a population mean
- test claims about the population mean using results from sampling
- recognise whether an alternative hypothesis leads to a one-tail or a two-tail test
- compare the approaches of using confidence intervals and hypothesis tests.

3.1.3 Level 2

The material at this level builds on Core Level 1. The material is now advanced enough for simple real engineering problems to be addressed. The material in this level can no longer be regarded as essential for every engineer (hence the omission of 'Core' from the title of this level). Different disciplines will select different topics from the material outlined here. Furthermore, different disciplines may well select different amounts of material from Level 2. Those engineering disciplines which are more mathematically based, such as electrical and chemical engineering, will require their students to study more Level 2 topics than other disciplines, such as manufacturing and production engineering which are less mathematically-based.

3.1.3.1 Analysis and Calculus

The material in this section covers the basic development of analysis and calculus consequent on the material in Core Level 1.

Ordinary differential equations

As a result of learning this material you should be able to

- understand how rates of change can be modelled using first and second derivatives
- recognise the kinds of boundary condition which apply in particular situations
- distinguish between boundary and initial conditions
- distinguish between general solution and particular solution
- understand how existence and uniqueness relate to a solution
- classify differential equations and recognise the nature of their general solution
- understand how substitution methods can be used to simplify ordinary differential equations
- use an appropriate software package to solve ordinary differential equations.

First order ordinary differential equations

As a result of learning this material you should be able to

- recognise when an equation can be solved by separating its variables
- obtain general solutions of equations by applying the method
- obtain particular solutions by applying initial conditions
- recognise the common equations of the main areas of application
- interpret the solution and its constituent parts in terms of the physical problem
- understand the term ‘exact equation’
- obtain the general solution to an exact equation
- solve linear differential equations using integrating factors
- find and interpret solutions to equations describing standard physical situations
- use a simple numerical method for estimating points on the solution curve.

Second order equations - complementary function and particular integral

As a result of learning this material you should be able to

- distinguish between free and forced oscillation

- recognise linear second-order equations with constant coefficients and how they arise in the modelling of oscillation
- obtain the types of complementary function and interpret them in terms of the model
- find the particular integral for simple forcing functions
- obtain the general solution to the equation
- apply initial conditions to obtain a particular solution
- identify the transient and steady-state response
- apply boundary conditions to obtain a particular solution, where one exists
- recognise and understand the meaning of ‘beats’
- recognise and understand the meaning of resonance.

<u>Functions of several variables</u>

As a result of learning this material you should be able to

- define a stationary point of a function of several variables
- define local maximum, local minimum and saddle point for a function of two variables
- locate the stationary points of a function of several variables
- obtain higher partial derivatives of simple functions of two or more variables
- understand the criteria for classifying a stationary point of a function of two variables
- obtain total rates of change of functions of two variables
- approximate small errors in a function using partial derivatives.

<u>Fourier series</u>

As a result of learning this material you should be able to

- understand the effects of superimposing sinusoidal waves of different frequencies
- recognise that a Fourier series approximation can be derived by a least squares approach
- understand the idea of orthogonal functions
- use the formulae to find Fourier coefficients in simple cases
- appreciate the effect of including more terms in the approximation
- interpret the resulting series, particularly the constant term
- comment on the usefulness of the series obtained.
- state the simplifications involved in approximating odd or even functions
- sketch odd and even periodic extensions to a function defined on a restricted interval
- obtain Fourier series for these extensions
- compare the two series for relative effectiveness
- obtain a Fourier series for a function of general period.

<u>Double integrals</u>

As a result of learning this material you should be able to

- interpret the components of a double integral
- sketch the area over which a double integral is defined

- evaluate a double integral by repeated integration
- reverse the order of a double integral
- convert a double integral to polar co-ordinates and evaluate it
- find volumes using double integrals.

Further multiple integrals

As a result of learning this material you should be able to

- express problems in terms of double integrals
- interpret the components of a triple integral
- sketch the region over which a triple integral is defined
- evaluate a simple triple integral by repeated integration
- formulate and evaluate a triple integral expressed in cylindrical polar co-ordinates
- formulate and evaluate a triple integral expressed in spherical polar co-ordinates
- use multiple integrals in the solution of engineering problems.

Vector calculus

As a result of learning this material you should be able to

- obtain the gradient of a scalar point function
- obtain the directional derivative of a scalar point function and its maximum rate of change at a point
- understand the concept of a vector field
- obtain the divergence of a vector field
- obtain the curl of a vector field
- apply simple properties of the operator ∇
- know that the curl of the gradient of a scalar is the zero vector
- know that the divergence of the curl of a vector is zero
- define and use the Laplacian operator ∇^2 .

Line and surface integrals, integral theorems

As a result of learning this material you should be able to

- evaluate line integrals along simple paths
- apply line integrals to calculate work done
- apply Green's theorem in the plane to simple examples
- evaluate surface integrals over simple surfaces
- use the Jacobian to transform a problem into a new co-ordinate system
- apply Gauss' divergence theorem to simple problems
- apply Stokes' theorem to simple examples.

Linear optimisation

As a result of learning this material you should be able to

- recognise a linear programming problem in words and formulate it mathematically
- represent the feasible region graphically
- solve a maximisation problem graphically by superimposing lines of equal profit
- carry out a simple sensitivity analysis
- represent and solve graphically a minimisation problem
- explain the term ‘redundant constraint’
- understand the meaning and use of slack variables in reformulating a problem
- understand the concept of duality and be able to formulate the dual to a given problem.

The simplex method

As a result of learning this material you should be able to

- convert a linear programming problem into a simplex tableau
- solve a maximisation problem by the simplex method
- interpret the tableau at each stage of the journey round the simplex
- recognise cases of failure
- write down the dual to a linear programming problem
- use the dual problem to solve a minimisation problem.

Non-linear optimisation

As a result of learning this material you should be able to

- solve an unconstrained optimisation problem in two variables
- use information in a physically-based problem to help obtain the solution
- use the method of Lagrange multipliers to solve constrained optimization problems
- solve practical problems such as minimising surface area for a fixed enclosed volume or minimising enclosed volume for a fixed surface area.

Laplace transforms

As a result of learning this material you should be able to

- use tables to find the Laplace transforms of simple functions
- use the property of linearity to find the Laplace transforms
- use the first shift theorem to find the Laplace transforms
- use the ‘multiply by t‘ theorem to find the Laplace transforms
- obtain the transforms of first and second derivatives
- invert a transform using tables and partial fractions
- solve initial-value problems using Laplace transforms
- compare this method of solution with the method of complementary function / particular integral.
- use the unit step function in the definition of functions

- know the Laplace transform of the unit step function
- use the second shift theorem to invert Laplace transforms
- obtain the Laplace transform of a periodic function
- know the Laplace transform of the unit impulse function
- obtain the transfer function of a simple linear time-invariant system
- obtain the impulse response of a simple system
- apply initial-value and final-value theorems
- obtain the frequency response of a simple system.

<u>z transforms</u>

As a result of learning this material you should be able to

- recognise the need to sample continuous-time functions to obtain a discrete-time signal
- obtain the z transforms of simple sequences
- use the linearity and shift properties to obtain z transforms
- know the 'multiply by a^k ' and 'multiply by k^n ' theorems
- use the initial-value and final-value theorems
- invert a transform using tables and partial fractions
- solve initial-value problems using z transforms
- compare this method of solution with the method Laplace transforms.

<u>Complex functions</u>

As a result of learning this material you should be able to

- define a complex function and an analytic function
- determine the image path of a linear mapping
- determine the image path under the inversion mapping
- determine the image path under a bilinear mapping
- determine the image path under the mapping $w = z^2$
- understand the concept of conformal mapping and know and apply some examples
- verify that a given function satisfies the Cauchy-Riemann conditions
- recognise when complex functions are multi-valued
- define a harmonic function
- find the conjugate to a given harmonic function.

<u>Complex series and contour integration</u>

As a result of learning this material you should be able to

- obtain the Taylor series of simple complex functions
- determine the radius of convergence of such series
- obtain the Laurent series of simple complex functions
- recognise the need for different series in different parts of the complex plane
- understand the terms 'singularity', pole

- find the residue of a complex function at a pole
- understand the concept of a contour integral
- evaluate a contour integral along simple linear paths
- use Cauchy's theorem and Cauchy's integral theorem
- state and use the residue theorem to evaluate definite real integrals

Introduction to partial differential equations

As a result of learning this material you should be able to

- recognise the three main types of second-order linear partial differential equations
- appreciate in outline how each of these types is derived
- state suitable boundary conditions to accompany each type
- understand the nature of the solution of each type of equation.

Solving partial differential equations

As a result of learning this material you should be able to

- understand the main steps in the separation of variables method
- apply the method to the solution of Laplace's equation
- interpret the solution in terms of the physical problem.

3.1.3.2 Discrete Mathematics

The material in this section covers the basic development of discrete mathematics consequent on the material in Core Level 1.

Number systems

As a result of learning this material you should be able to

- recognise the Peano axioms
- carry out arithmetic operations in the binary system
- carry out arithmetic operations in the hexadecimal system
- use Euclid's algorithm for finding the greatest common divisor

Algebraic operations

As a result of learning this material you should be able to

- understand the notion of a group
- establish the congruence of two numbers modulo n
- understand and carry out arithmetic operations in Z_n , especially in Z_2
- carry out arithmetic operations on matrices over Z_2
- understand the Hamming code as an application of the above (any other suitable code will serve just as well).

Recursion and difference equations

As a result of learning this material you should be able to

- define a sequence by a recursive formula
- obtain the general solution of a linear first-order difference equation with constant coefficients
- obtain the particular solution of a linear first-order difference equation with constant coefficients which satisfies suitable given conditions
- obtain the general solution of a linear second-order difference equation with constant coefficients
- obtain the particular solution of a linear second-order difference equation with constant coefficients which satisfies suitable given conditions

Relations

As a result of learning this material you should be able to

- understand the notion of binary relation
- find the composition of two binary relations
- find the inverse of a binary relation
- understand the notion of a ternary relation
- understand the notion of an equivalence relation on a set
- verify whether a given relation is an equivalence relation or not
- understand the notion of a partition on a set
- view an equivalence either as a relation or a partition
- understand the notion of a partial order on a set
- understand the difference between maximal and greatest element, and between minimal and smallest element.

Graphs

As a result of learning this material you should be able to

- recognise an Euler trail in a graph and / or an Euler graph
- recognise a Hamilton cycle (path) in a graph
- find components of connectivity in a graph
- find components of strong connectivity in a directed graph
- find a minimal spanning tree of a given graph.

Algorithms

As a result of learning this material you should be able to

- understand when an algorithm solves a problem
- understand the 'big O' notation for functions
- understand the worst case analysis of an algorithm

- understand one of the sorting algorithms
- understand the idea of depth-first search
- understand the idea of breadth-first search
- understand a multi-stage algorithm (for example, finding the shortest path, finding the minimal spanning tree or finding maximal flow)
- understand the notion of a polynomial-time-solvable problem
- understand the notion of an NP problem (as a problem for which it is "easy" to verify an affirmative answer)
- understand the notion of an NP-complete problem (as a hardest problem among NP problems).

3.1.3.3 Geometry

The material in this section covers the basic development of geometry consequent on the material in Core Level 1.

<u>Helix</u>

As a result of learning this material you should be able to

- recognise the parametric equation of a helix
- derive the main properties of a helix, including the equation of the tangent at a point, slope and pitch.

<u>Geometric spaces and transformations</u>

As a result of learning this material you should be able to

- define Euclidean space and state its general properties
- understand the Cartesian co-ordinate system in the space
- apply the Euler transformations of the co-ordinate system
- understand the polar co-ordinate system in the plane
- understand the cylindrical co-ordinate system in the space
- understand the spherical co-ordinate system in the space
- define Affine space and state its general properties
- understand the general concept of a geometric transformation on a set of points
- understand the terms 'invariants' and 'invariant properties'
- know and use the non-commutativity of the composition of transformations
- understand the group representation of geometric transformatoions
- classify specific groups of geometric transformations with respect to invariants
- derive the matrix form of basic Euclidean transformations
- derive the matrix form of an affine transformation
- calculate coordinates of an image of a point in a geometric transformation
- apply a geometric transformation to a plane figure.

3.1.3.4 Linear Algebra

The material in this section covers the basic development of linear algebra consequent on the material in Core Level 1.

Matrix methods

As a result of learning this material you should be able to

- define a banded matrix
- recognise the notation for a tri-diagonal matrix
- use the Thomas algorithm for solving a system of equations with a tri-diagonal coefficient matrix
- partition a matrix
- carry out addition and multiplication of suitably-partitioned matrices
- find the inverse of a matrix in partitioned form.

Eigenvalue problems

As a result of learning this material you should be able to

- interpret eigenvectors and eigenvalues of a matrix in terms of the transformation it represents
- convert a transformation into a matrix eigenvalue problem
- find the eigenvalues and eigenvectors of 2x2 and 3x3 matrices algebraically
- determine the modal matrix for a given matrix
- reduce a matrix to diagonal form
- reduce a matrix to Jordan form
- state the Cayley-Hamilton theorem and use it to find powers and the inverse of a matrix
- understand a simple numerical method for finding the eigenvectors of a matrix
- use appropriate software to compute the eigenvalues and eigenvectors of a matrix
- apply eigenvalues and eigenvectors to the solution of systems of linear difference and differential equations
- understand how a problem in oscillatory motion can lead to an eigenvalue problem
- interpret the eigenvalues and eigenvectors in terms of the motion
- define a quadratic form and determine its nature using eigenvalues.

3.1.3.5 Statistics and Probability

The material in this section covers the basic development of statistics and probability consequent on the material in Core Level 1.

One-dimensional random variables

As a result of learning this material you should be able to

- compare empirical and theoretical distributions
- apply the exponential distribution to simple problems
- apply the normal distribution to simple problems
- apply the Weibull distribution to simple problems
- apply the gamma distribution to simple problems.

Two-dimensional random variables

As a result of learning this material you should be able to

- understand the concept of a joint distribution
- understand the terms ‘joint density function’, ‘marginal distribution functions’
- define independence of two random variables
- solve problems involving linear combinations of random variables
- determine the covariance of two random variables
- determine the correlation of two random variables.

Small sample statistics

As a result of learning this material you should be able to

- realise that the normal distribution is not reliable when used with small samples
- use tables of the t-distribution
- solve problems involving small-sample means using the t-distribution
- use tables of the F-distribution
- use pooling of variances where appropriate
- use the method of pairing where appropriate.

Small sample statistics: chi-square tests

As a result of learning this material you should be able to

- use tables for chi-squared distributions
- decide on the number of degrees of freedom appropriate to a particular problem
- use the chi-square distribution in tests of independence (contingency tables)
- use the chi-square distribution in tests of goodness of fit.

Analysis of variance

As a result of learning this material you should be able to

- set up the information for a one-way analysis of variance
- interpret the ANOVA table
- solve a problem using one-way analysis of variance
- set up the information for a two-way analysis of variance
- interpret the ANOVA table
- solve a problem using two-way analysis of variance.

Simple linear regression

As a result of learning this material you should be able to

- derive the equation of the line of best fit to a set of data pairs
- calculate the correlation coefficient

- place confidence intervals around the estimates of slope and intercept
- place confidence intervals around values estimated from the regression line
- carry out an analysis of variance to test goodness of fit of the regression line
- interpret the results of the tests in terms of the original data
- describe the relationship between linear regression and least squares fitting.

Multiple linear regression and design of experiments

As a result of learning this material you should be able to

- understand the ideas involved in a multiple regression analysis
- appreciate the importance of experimental design
- recognise simple statistical designs.

3.1.4 Level 3

This level is the one at which the mathematical techniques covered should be applied to a range of problems encountered in industry by practising engineers. These advanced methods build on the foundations laid by Levels 1 and 2 of the curriculum. It is quite possible that much of this material will be taught not within the context of dedicated mathematical units but as part of units on the engineering topics to which they directly apply. It is expected that significant use will be made of industry standard mathematical software tools. The specialised nature of these techniques and the importance of their application in an engineering setting makes detailed learning outcomes (as given for the other levels of the curriculum) less straightforward to define. For this reason only a list of general topic headings will be given. This material will be taught only towards the end of a degree programme.

3.1.4.1 Analysis and calculus

- Numerical solution of ordinary differential equations
- Fourier analysis
- Solution of partial differential equations, including the use of Fourier series
- Fourier transforms
- Finite element method

3.1.4.2 Discrete mathematics

- Combinatorics
- Graph theory
- Algebraic structures
- Lattices and Boolean algebra
- Grammars and languages

3.1.4.3 Geometry

- Differential geometry
- Geometric modelling of curves and surfaces
- Geometric methods in solid modelling
- Non-Euclidean geometry
- Computer geometry
- Fractal geometry

- Geometric core of Computer Graphics

3.1.4.4 Linear Algebra

- Matrix decomposition
- Further numerical methods

3.1.4.5 Statistics and probability

- Stochastic processes
- Statistical quality control
- Reliability
- Experimental design
- Queueing theory and discrete simulation
- Filtering and control
- Markov processes and renewal theory
- Statistical inference
- Multivariate analysis

3.1.4.6 Other subjects

- Chaos theory
- Fuzzy mathematics

3.2 Overarching themes

In order to foster "overarching" sense making, in the OECD PISA and other documents the content-related competencies have not been organized according to the traditional areas of mathematics but rather along some general themes, called "overarching ideas" in (OECD 2009). The ideas stated there are "quantity", "space and shape", "change and relationships", and "uncertainty" (OECD 2009, p. 93-104). In the following, we reuse themes stated in (Ministry of Education Baden-Württemberg 2004) for the grades 6 to 12 and add some new ones that are specific to engineering education. These themes can be used by a lecturer for offering a wider perspective to the students by embedding a specific mathematical topic into an overarching mathematical context. They can also be used in projects where students investigate the basic concepts within a theme and their extension by using more advanced concepts.

Quantity

Several mathematical topics and learning outcomes are related to the concept of quantity. Students should know the natural, integer, rational and real numbers as a means of quantification mainly from school education although some properties of the reals may also be dealt with in tertiary education. In university education, the means of quantification and their handling and understanding are extended in several ways. The set of complex numbers is introduced as extension of the reals where "more" operations are possible; moreover, complex numbers "store more information" since they contain two items of real numbers and they can be visualized as points in the complex plane. This is used to store amplitude and phase information of a vibration in one entity. Another extension are vectors as a concept for modelling quantities which cannot just be described by one number since they also have a direction; for handling these, operations are defined which are partially similar to those well-known from numbers, partially specific to vectors. Thirdly, the ubiquitous work with computers makes it necessary to be aware of the restrictions of computer-representable numbers and the corresponding numerical problems when performing operations with these.

The content-related learning outcomes stated in the previous section which can be subsumed under the theme “quantity” are to be found in sections 3.1.1.1, 3.3.3.2, 3.1.2.1, 3.1.2.4, and 3.1.3.2.

Measuring

In school education students learn how to measure elementary geometric objects (area of a triangle, volume of a cylinder). This is extended in university education in several ways. Geometric objects can also be measured when they have curved boundary curves or surfaces by considering them as “infinite sum of infinitely small quantities” (using simple and multiple integrals). This is particularly important in engineering where curved boundaries occur very often. Moreover, the measurement is not restricted to measuring geometric properties but also application quantities like moment of inertia or electric charge. The measurement can also be done in an approximative way when discrete data representations of the curved objects are used. Errors should be estimated in this case. Finally, measurements can also be done with time-dependent data (like signals) from which meaningful information (e.g. frequency content) can be computed either symbolically or again (approximatively) numerically.

Space and shape

Engineering students have to extend their understanding of space to multi-dimensional vector spaces and recognize which properties of the three dimensional real space are retained.

Regarding shape, formal mathematical description is no longer restricted to a set of simple geometries (lines, surface, bodies) but can be extended to arbitrary geometries (free form geometries) which can hence be made available for computation. Such geometries cannot only be analyzed but also be designed in order to fulfill specific shape requirements. Students should know the possibilities of creating and fixing shape properties by applying operations and specifying relations and be able to apply these to create easily changeable geometric objects in CAD systems.

Functional dependency

The understanding of functional dependency acquired at school is extended to functional dependency of multidimensional quantities (e.g. several independent and/or dependent quantities). Possibilities and restrictions of transferring concepts and properties from the one-dimensional to the multi-dimensional case need to be recognized. These multidimensional dependencies are needed in several application scenarios (e.g. force vectors varying with point in space; formulae containing several variable parameters). Linear dependencies between several variables are often described in text books using matrices, and manipulations and computations are performed within the matrix calculus. A further extension of the theme “functional dependency” is the dependency between complex quantities. This is often useful in engineering for a unified and simplified treatment. For example, in the description and design of planar motion, complex-valued time-dependent functions are used.

Another additional aspect of the theme of functional dependency is the design of such dependencies in different engineering areas. For example, in motion design the piecewise design of functions is a frequent task, and in control theory the design of optimal control functions is quite important.

Dependencies between functions

In addition to functional dependency the dependency and relationship between functions often comes up in engineering applications. There are relationships between functions and their derivatives (rates of change) leading to differential equations. There are relationships between functions which can be understood as transformations (building derivative and integral, functional transformations like Laplace or Fourier). Moreover, functions can be considered as finite and infinite combinations of basic functions (power functions, trigonometric functions) which allows to simplify operations (by doing them for each summand separately) or to extract useful information (e.g. on frequencies).

Data and chance

Handling of data, deterministic or based on random processes, is ubiquitous in engineering. Often data have to be fitted to mathematical models by using interpolation or approximation methods. Quantities with values influenced by chance are modeled as random variables and adequate probability models based on properties of the quantity must be found. The quality of probability models and key figures computed from random data has to be judged using statistical tests and confidence intervals.

Algorithms

Algorithms as methods for computing mathematical objects (solution of an equation, derivative, integral etc.) are important for getting an understanding of what kind of problems can be solved when working within a mathematical model. There are symbolic algorithms and numerical algorithms, both having their specific advantages and disadvantages which need to be known for making an adequate choice of what to use to tackle a specific problem.

There is also a distinction between algorithms applying heuristic improvement strategies (based on a formal computational model) and those which optimize the value of a variable.

Modelling

Modelling application behavior is a general theme in engineering, application text books are full of mathematical models. These comprise functional dependency models (e.g. for growth processes, for saturation behavior, for periodic behavior, dependency between vectors), equation models (e.g. for describing equilibrium in statics), differential equation models (e.g. dynamic equilibrium of forces, relationship between time-dependency and dependency on location in PDEs), differential models (models constructed using “infinitely small objects” as limits of finitely small objects as in integrals for moments of inertia). Engineers should recognize the same mathematical model in different applications and value the advantage of abstraction and reuse of formalism, knowledge of properties and algorithms for computing interesting quantities. Engineers should be able to formulate application problems mathematically and use mathematical methods for solving them.

Structures

Structures are a fundamental theme in mathematics. Many engineering situations have common structures and patterns and their recognition is useful for reducing the necessary mathematical models to a manageable number of types. Insights gained in one situation can be reused in others, the same “language” can be used. These essential structures include the following:

- number sets as structure (with objects, operations, properties)
- real and complex vector spaces as linear structures (objects, operations, properties); use of such structures for describing solution spaces in linear models (equations, differential equations)
- matrix spaces as structure (objects, operations properties)
- structures which are build from basic “objects” (e.g. fractions built from partial fractions; periodic functions built from sine functions; functions built from powers; vectors as linear combination of eigenvectors; motion built from rotation and translation); use of structural knowledge for reducing complex (strong) problems to easier ones and for recognizing properties.

4 Teaching and learning environments

Here comes a list with different teaching and learning settings like lecture, exercise sessions/tutorials, home assignments with and without technology, individual and group projects with documentation and presentation, problem-based learning, common teaching with application colleagues, distance learning; activation, use of technology, integration of mathematics and application subjects are dealt with here.

Here, we can also write about learning environments which address the transition problem (support centers, foundation years, additional classes in school)

References: (Gavalcova 2008), (Bergsten 2007), (Christensen 2008), (Niss 2001), (Challis 2010), (Clements 2000), Robinson, Lawson and others on transition aspects, e.g. (Lawson 2004b)

4.1 Teaching and learning arrangements

To be provided

4.2 Transition issues

To be provided

4.3 Technology issues

Any university education should consider that students must be taught in conditions that favour their integration into larger society with respect to its particular characteristics, including all recent technological advancement. The effect of computer technology on education seems to be greater in mathematics than in any other subject. Close links between the two disciplines might be a consequence of the fact that computer science was a part of mathematics at the beginning, while later it has developed into an independent wide-ranging technical discipline. The term "technology", though, is very comprehensive and includes a lot of different artifacts, for example:

- Pocket calculators with different symbolic and/or numerical and/or graphical capabilities
- Mathematical computer programmes (symbolical and numerical ones, e.g. CAS, Matlab, Spreadsheet)
- Engineering programmes based on mathematical models which "shine through" to a certain extent (CAD, FEM, mechanism design, multi-body dynamics, CFD, …)
- Mathematical material available online (like Wikipedia) not specifically dedicated to serve instructional purposes
- Special instructional websites or stand-alone programmes, also for practicing mathematics
- Technology supporting the communication in the learning process (e-mail forum, voting system)
- Assessment support programmes (including self-assessment).

Computer visualisations and modelling, computational capacity of available computer algebra systems and rapidly expanding applications of mathematical models in many different areas, technical sciences in particular, have gradually influenced mathematical education and curricula and have led to a new specific character of teaching mathematics in different teaching and learning settings and environments. The future engineers will face many problems that cannot be solved using classical methods, or this might be excessively time consuming and increasing costs, which would often simply not be possible, e.g. in the control and optimisation of a particular industrial process. In order to specify the bases of the programs that will solve practical engineering problems, the engineer must be able to handle the mathematical model of the process. The full comprehension of the mathematical theory is demanded rather than skill to solve the specific problem itself. Such accumulation of the necessary mathematical knowledge of a future engineer has to be performed consistently and in various ways, by using computers from the very first year of the engineering study. On the other hand, a deep mathematical conceptual understanding is required when computers are used for solving non-elementary mathematical tasks. How to settle the proper proportion balancing the traditional teaching and learning methods with the application of ICT in different environments remains as an everlasting topic for discussions.

There exist various ways how ICT can be utilized in maths education:

- Direct methods - usage in educational process and assessments
    - CAS as powerful demonstration tool for lecturer
    - CAS for exploring and calculations during maths lessons
    - Pocket / graphing calculators
    - CAS utilised by students in solving mathematical problems and projects
    - E-learning solutions for maths courses
    - Computer based assessment
- Indirect methods – additional study resources for self-study and evaluation
    - On-line maths courses
    - On-line maths databases with electronic resources
    - Mathematics support centres on-line
    - Usage of web-based materials
    - Tutorials by e-mails
    - Maths chats on Internet

Adapting the mathematical educational process to new ICT tools one must be aware of the risk that this computer-based learning environment may cause an unexpected transmission of the bases of "traditional mathematical culture". This is not just an efficient mathematical practice supported by currently available computational tools, but rather a stable amount of core mathematical knowledge and ideas, having a strong impact on the overall reasoning, that was always considered by engineers as natural. Most students love using a computer. Just having a computer involved can make a huge difference in students' attitude and feelings towards mathematics. However, the special facilities provided by the computers need new approaches in teaching mathematical subjects. In some topics computer animation can greatly increase the effectiveness of the teaching process, which is especially true e.g. in calculus or multivariable calculus, in the theory of differential equations or in geometry, where CAS (Computer Algebra Systems) can be used as demonstration and visualization tools for better conceptual understanding. CAS are cognitive tools, as they facilitate the technical dimension of mathematical activity and allow the user to take action on mathematical objects or representations of those objects. CAS helps to make especially difficult abstract concepts more accessible and understandable to students. In addition, they help to increase motivation and to improve students' attitudes towards Mathematics. On the other hand, using CAS helps to bring about certain changes in the way classes are held, as their usage requires student active participation, autonomous activity and interaction among students making the process of acquiring and developing mathematical knowledge more student-centred. Due to the potential interactivity of these tools, students are able to attain a higher level of abstraction in mathematical problem-solving, something which clearly represents a significant didactic accomplishment.

Based on these arguments, some universities have fully integrated CAS into mathematics teaching for different university degrees, while their use is no longer considered to be innovative, but rather a common practice in some maths courses. However, we must also keep in mind the limitations and risks involved. Extensive and exclusive usage of CAS can potentially prevent students from making the proper connections between the techniques used for calculations and conceptual understanding or mental approach to Mathematics. It is also necessary to adopt a series of appropriate measures to help students know what to do when a CAS fails to give them an answer to a problem and to help them recognize when CAS are useful for solving a given problem and when they are not. Regardless, a common characteristic has been detected regarding the effects of working with CAS: despite the improvements in attained practical results, most students are not aware of the improvements in their knowledge and skills, nor are they aware of the improvements in their assimilation of the contents presented in class (J.L. Galán García, M.A. Galán García, A. Gálvez Galiano, A.J. Jiménez Prieto, Y. Padilla Domínguez and P. Rodríguez Cielos (2005)).

Computers nowadays play an important role not only as a calculation engine and visualization tool, but also as a communication media and virtual environment. Usage of ICT, e-learning instructional resources and Internet as communication tool in teaching and learning mathematics has been the main topic of several SEFI MWG seminars and has been discussed in many contributions. The latest development shows increasing demand on complex solutions, while in addition to the traditional university study programme schemes with settled contact hours, an additional mathematical support is provided to students via mathematical support centre, accessible both virtually and in present form. Integrating technology into mathematical education of engineers was also a topic for discussion groups at the MWG seminar in 2006 for which there is a special report (Alpers 2006).

Online student resources come in two types: online mathematics concept instruction and online practice. Some websites contain only instruction, some only practice, but many websites contain a combination of practice and instruction. Online Math instruction is often an example question with the answer and an explanation of how the answer was obtained. Math instruction online may also be in the form of a video or animation, with an audio explanation. This can be particularly helpful to students in the area of geometry, where dynamic mathematical applets can be used for interactive demonstration of particular relations. The other benefit of online instruction is that the student can view the instruction again and again without embarrassment in front of peers and without any frustration on the part of the teacher. The neutrality of a computer teacher makes all the difference to students who may have anger management issues or problems with authority figures, or who might perceive discrimination due to social and gender stereotypes.

Using computers for assessment still remains a rather questionable and risky idea, though examples of a good practise exist, as e.g. proposal for a combination of an open source learning platform together with an open source computer algebra system for generation of custom-made exercises by CAS-script assessing the answers is given in Risse (2008). Analysis of the usage of interactive online mathematics testing and teaching system for the first year students can be found in Lehtonen (2008).

Engelbrecht and Harding (2005a, pp. 244-245) developed the following general descriptors, formulating six mathematics-specific categorisations for on-line mathematics courses.

- *Dynamics and Access* - What is the frequency of access necessary for success in the course?
- *Assessment* - How much of the assessment is done on site?

- *Content* - How much of the course content is on the site?
- *Communication* - How much communication happens on-line?
- *Richness* - How many enriching components does the site have?
- *Independence* - How independent from face-to face contact is success in the course?

These might help in measuring the level of ICT integration into the mathematics education and its impact on development of new learning settings and environments and can be regarded as first steps to introduce measurements of technology use in general introducing a consistent taxonomy.

Most technical university courses of mathematics naturally integrated the ICT usage into the mathematics curriculum; course syllabi include specific statements about the appropriate use of technology in the particular topics. These usually reflect the major benefits of technology, as multiple dynamic representation, calculation and demonstration power or modeling and simulation tool, emphasize the added value achieved by the effective implementation of technology and advice teachers to use it. They demonstrate the impact of the recent society behavioral change in attitudes caused by the technical development, and reveal the needs to consider the cognitive and epistemological dimensions of on-line mathematical education in the 21st century in a wider scope of integration issues considered by the taxonomy.

The taxonomy and the other findings of the thesis Oates G. (2009) may meet the challenge as author provides a means by which the full potential of an *Integrated Technology Mathematics Curriculum* can be revealed and exploited. Six characteristics presented in the table below are defined in order to describe broadly the extent of technology use in a course, which are proposed as an initial taxonomy for an integrated-technology course.

Characteristics		Example of questions asked to examine the degree of integration
A	Access	To what extent do students have access to technology tools, e.g. is it compulsory? Do they use their own, or access it in computer labs?
B	Student Facility	How proficient are students with the use of the technology, and what assistance is provided to help them?
C	Assessment	Is technology expected and/or permitted in assessment?
D	Pedagogy	How and when do the staff and students interact with the technology? For example, is it used mainly as a complex calculation device and demonstration tool, or to develop and explain concepts?
E	Curriculum	Has the course curriculum, for example content, order of teaching, changed to reflect the use of technology?
F	Staff Facility	Are staff familiar with the use and capabilities of the technology, both mathematically and pedagogically?

A Taxonomy for Integrated Technology Characteristic

Some investigations have also been carried on in connection to the students´ approach, reactions and their attitude to the utilisation of new technologies. There is strong evidence that students enjoy well designed web-based learning resources, anyhow ill-considered computer aided teaching and assessment may produce frustration and anxiety. Changing role of teachers, who become more tutors and instructors than lecturers, seems to be evident, see the analysis of the feedback from students and their opinion to the on-line Pilot course in Differential and Integral Calculus in Velichová et al. (2004), or evaluation of students´ reaction to the project on utilising graphing calculators in teaching linear algebra at secondary schools, which can be found in Verweij (2004).

Finally technology issues can be meaningfully connected to the competence approach. In the sequel, we state how the mathematical competencies stated in chapter 2 are affected by the introduction of technology in the mathematical education of engineers:

- Thinking mathematically: This competency should also include the ability to recognize that for a problem at hand there should be a programme based on a mathematical algorithm or that it should be possible to implement one's own routine within a certain technical environment.
- Reasoning mathematically: Technology allows an experimental working style where one makes variations, for example to investigate the influence of parameters or to find parameter configurations such that a certain property is achieved. At first sight, try and error ("empirical plausibility") seems to replace the need for mathematical reasoning but in a huge design space it is still important to apply mathematical reasoning (e.g. influence of symbols within a formula) in order to make variations efficiently. Technology also allows to perform simulation experiments to find patterns or find counter-examples for assumptions. Still students should know the difference between proof and experimental plausibility.
- Posing and solving mathematical problems: Technology allows for an experimental problem solving style using heuristics, using knowledge about the influence of factors and using already implemented numerical solution procedures. With technology learners can set up their own experimentation environment to get answers.
- Modelling mathematically: Technology allows to work with more complex and realistic models, since work within the model (e.g. easy variation) is supported by technology, sometimes even the setup of models. In engineering programmes, models are only partially visible, so students need to work with technology where the underlying model is not exactly known. This requires knowledge about strategies for checking one's understanding of the workings of the programme and also for checking the results.
- Representing mathematical entities: Technology provides new representations which can be used as a cognitive aid for understanding a mathematical concept (e.g. a 3D plot which can be rotated; other representations in CAD systems). In particular, dynamic representations are available which in former times were only possible by constructing mechanical devices. Moreover, the possibility of interactive manipulation of representations supports the exploration of relations between different representations.
- Handling mathematical symbols and formalism: Programmes and mathematical material still require mathematical symbols and formalism as input, sometimes with a programme-specific syntax, sometimes facilitated by pallet-style input. The same holds true for the output which

needs to be understood. The tedious handling of extensive routines can be delegated to programmes.

- Communicating in, with, and about mathematics: Mathematical programmes can be used as means for communication, when the user documents and presents solutions to problems within the programme (e.g. CAS or spreadsheet). For this the user has to encode the mathematical ideas, objects and procedures with the expressive means the programme offers. Students also have to decode such documentations when they have been set up by others. They also have to find and understand mathematical material on the web and to communicate mathematical solutions or ideas on the web (often not having the possibility to draw and verbally explain things as is possible in face-to-face communication).
- Making use of aids and tools: Students should be able to retrieve relevant information from material available on the web, they should have a sense of what to expect from such material. When they use programmes they should know about their capabilities and limitations, and they should able to check the plausibility of programme output.

The relationship between technology usage and competence acquisition has (at least) two facets: On the one hand using technology can help in the acquisition of competencies, on the other hand does knowledgeable technology usage require special additional aspects of each competency.

4.4 Connections between mathematics and application subjects

To be provided

4.5 Attitudes

If – as proposed in this document – it is the ultimate goal of mathematical education of engineers to make them mathematically competent, and if this competence is defined as "insightful readiness to act in response to *a certain kind of mathematical challenge* of a given situation" (Blomhoj & Jensen 2007, p.47), then such readiness is strongly related to the attitude a student has towards mathematics. In their studies on attitudes of engineering students to mathematics in a few British universities, Shaw & Shaw (1997, 1999) found out that only about one third of the students were motivated, about 75% had the desire to improve their mathematical abilities and a broad range from 20% to 66% perceived mathematics as being difficult. According to the authors, the attitude of students towards mathematics is more positive when the environment provided by universities is perceived as being supportive. This can be achieved by organizing support in additional tutorials, foundation classes, online materials or mathematical support centers, for example (cf. section 4.2 of this document). This can at least prevent students with difficulties in mathematics to turn into mathematics "haters" (Shaw & Shaw 1999).

Booth (2004, p. 18) investigated the different perceptions of mathematics with engineering students in more detail and distinguished between three views of mathematics:

- Mathematics is "a part of the degree programme".
- Mathematics is a "basis for other subjects".
- Mathematics is a "tool for analyzing problems that occur in the world …".

In the first view, mathematics is just an "isolated subject", whereas in the second it is "integrated into the programme of study" and in the third view also "into the world it describes". These different kinds of perception have a considerable influence on the students' view of their own responsibility for the learning process and their approach to mathematical learning. Booth distinguishes between a “surface approach” where students focus on the “sign”, on the demands of the course and on the reproduction of course material, and a “deep approach” where students focus on meaning, construct relations between mathematics and engineering subjects and also to their wider experience. If a student sees mathematics (or a certain part of mathematics since this can differ from topic to topic) as an isolated subject he is likely to apply a surface approach to learning whereas a view relating mathematics to other subjects and the world will rather lead to a deep approach. Therefore, inducing in students a realistic perspective of the role of mathematics in the study programme as well as in later engineering life is important for achieving a deep learning approach.

In her investigation of the “Mathematical Disposition of Structural Engineers”, Gainsburg (2007) proposes that mathematical education should strive for a similar “mathematical disposition” as she found with structural engineers and which she termed “skeptical reverence”: “mathematics is a powerful and necessary tool that must be used judiciously and skeptically” (p. 498). One could denote such an attitude also as critical appreciation: Mathematics can be of help in many engineering situations but it is not the only constituent of engineering work since there are many other aspects to be taken into account which are different from those that can be stated and treated in a mathematical way.

In general mathematics education, the topic of attitude is discussed under the heading of “beliefs and affects” (Schoenfeld 1992, Cardella 2008). Here, it is also emphasized that the beliefs about mathematics are largely shaped by the experience in school education and that – on the other hand – these beliefs shape the mathematical behavior shown by students. Therefore, it is quite important to create experiences where mathematical thinking is seen as a process which helps in capturing and solving real problems and not just a five-minute activity to work on an isolated exercise. Only by having made such experience will the students in their later engineering life be willing to use mathematical thinking for solving their engineering problems.

How can mathematical education encourage and strengthen a perception of mathematics which is integrated in the engineering world (be it educational or a real work environment) and where mathematics is critically appreciated as relevant part of the problem solving process? The competence approach already intends to make students see what mathematics can do for them (mathematical thinking) and it emphasizes the “action character” and the contextualization since students should be enabled to master the mathematical challenges they meet in engineering contexts. Therefore, the competence approach seems to be particularly suitable for creating and supporting a desirable attitude towards mathematics. There will still be large differences regarding mathematical abilities but having a good understanding of what mathematics can do in engineering contexts and a realistic perception of own abilities (What can I do myself, where do I need an expert?) should lead to a realistic und helpful attitude for a professional engineer.

5 Assessment

Ever since John Biggs published the book “Teaching for Quality Learning at University” in 1999 (Biggs 1999), the idea behind the book, constructive alignment, has had an increasing impact on the teaching – learning – assessing cycle at many universities.

One guideline in constructive alignment, there is of course a lot more to it, is that the planning of a course or a module must give answers to three questions: – What shall the students learn, what is the expected learning outcome? – What shall the students do to learn, what is the best way to organize the teaching and learning? – How can the students’ knowledge be evaluated, what forms of assessment are most suitable?

In this chapter we discuss the third question, that of assessment, and begin with an overview of the different forms of assessment that are in use around Europe and was identified in a SEFI MWG project, the assessment project, reported at the SEFI MWG seminar in Vienna 2004 (Lawson 2004a).

5.1 Forms of assessment

The most common assessment method is a written examination, with closed books, at the end of the course. Less common is a written examination with open books or computer facilities to support the problem-solving. There is an ongoing debate on the use of calculators in university mathematics. In secondary school this debate is since long ended; the graphics calculators are seen to be essential tools for mathematical modeling and experiments. At university level computers are used for these purposes. When it comes to written examination with closed books calculators are in general not allowed. If they are there is, or should be, restrictions on the types or models. No graphics, no CAS. Modern advanced calculators are more or less equivalent to open books and personal notes being allowed.

Another difference worth mentioning is the written exam duration. In some countries four or five hours are standard, in other only one or two. Every written exam is a spot-check but the shorter duration the less of the contents can be covered and the more it opens for gambling strategies. The shorter duration can be compensated, like in many central European institutions, by a follow-up oral examination. Either for all students or for those that scored well enough. Teachers comment on oral examination that it is highly staff intensive but give the best opportunity to test in-depth understanding.

Take away assignments are used at several institutions, but always as one amongst a number of methods of assessment and never as the only or primary method. They give students an opportunity to explore more realistic problems than they can in an ordinary written examination and for this reason often require the use of computer software to complete the assessment task. Some teachers have reservations about this method of assessment because it is impossible to be certain that the student submitting the work actually did it for him/her-self. When the take away assignment is followed up with an oral presentation of the work the legal certainty is stronger.

Only a few institutions use multiple choice tests and those that do use them do so only occasionally. Such tests can be cheap to administer as they can be computer delivered and marked. They can be useful in giving formative feedback during the course. There are reasons to believe that the use of this kind of assessment is increasing. We will discuss this in detail later in this chapter. Again it is impossible to be certain that the student submitting the work actually did it for him/her-self, unless the test is implemented under invigilation and on computers that are not connected to any net.

Furthermore, as all that is marked is the student's final answer, they have limitations when being used for summative assessment.

Other methods of assessment such as project work, group work and oral presentations are not widely used. However, when it comes to examination of mathematical competencies, these methods can be more interesting. It is difficult to give individual grading of group work, but individual time-logs, progress-logs and contribution-reports together with the project report can help.

5.2 Assessing competencies

Include examples showing how different competencies can be assessed.

5.3 Technology-supported assessment

The use of technology-support in mathematics has several reasons. One, which is not within the scope of this chapter, is to simplify or improve the students' work, like in modelling projects or problem solving when a mathematical model is developed, implemented, tested or simulated with support of computer algebra systems or computations. Most universities offer the students commercial software like Mathematica or Maple (CAS) and Matlab or the freeware Octave for computations. There are also many commercial or free programs designed for specific types of problems. The students have to spend considerable time to learn how to use the software and there is a strong need for support from teachers when they do so. In most engineering programmes the proficiency to use computer software in problem solving is an expected learning outcome.

Next we will discuss how technology may support formative assessment during the course and summative assessment after the course or course module.

In its most primitive setting a test suitable for a computer-supported assessment system consists of a number of multiple choice questions consisting of a question together with one correct answer and a number of incorrect answers, distractors. The distractors must be close enough to the correct answer. The student has to select the correct answer to all or most questions in order to pass or to get a positive feedback. The student's work is not simplified or improved; she could do the same with paper and pen, as long as the questions are similar to those in the textbook. The advantage for the teacher is that once the system is there and a suitable set of exercises or questions are imported to the system, the system will do the work. The advantage for the student is that she can often do the test anywhere and anytime. If the test is created by a randomized selection of questions out of a large question bank the student can do the test many times. She will get immediate feedback and the teacher will get immediate information about the student's progress. The need of multiple choice question decreases if the test system is supported by a computer algebra system. But instead there can be specific demands on how the answer is given or formulated. A correct answer in wrong format is considered to be an incorrect answer, confusing the student of course. One challenge for the teacher is to find questions that assess a deeper understanding of the subject and still has one correct answer. Another challenge is to rethink the assessment and find questions that could not be asked when only paper and pen were available.

The feedback to the student in a simple system consists only of a mark correct/incorrect; the student has to find out what to do to improve. A more advanced system includes also learning support for the students. We can see many reasons, and also attempts, however still visionary, to strive towards complete, computer supported systems, so called Intelligent Tutoring Systems (ITS), where the student gets information not only about his/her errors or mistakes but also about the underlying misconceptions or lack of knowledge together with support to fill the gaps. At the moment the student needs help from a human teacher to figure out the nature of the misconceptions and what to do to improve. Some of this can be e-support linked to the test; some may be personal given by the teacher or a support center on request from the student. The nature of the support can also vary from “read this example” or “view this explanation” to “read again chapter X in your textbook”.

There are specific problems with the legal certainty when computers are used in summative assessment or when students get some kind of credit (bonus points) for the performance in a formative assessment. In general the computers at universities are connected to a network and to internet. To prevent cheating the network connections must be closed, perhaps some other programs must be blocked and the students’ work must be invigilated. If the number of students is greater than the number of available computers, the need of randomized tests is obvious. The tests cannot be too similar otherwise the last students can have some unlawful help from the first. Still all tests must be of the same difficulty. If the test is done out of campus or out of office hours then the examiner does not know who actually took the test. For these reasons at most a minor part of the entire assessment should be computer-supported and not invigilated.

The paper “A review of computer-assisted assessment” by Gráinne Conole and Bill Warburton (Conole&Warburton 2005) gives a survey of both the use of technology for assessment and of the research on this use. There is an obvious need for a thorough survey of today’s use of technology for assessment in mathematics and a deep discussion concerning the consequences of that use.

5.4 Requirements for passing

In the first part of this chapter we recalled the findings of the SEFI MWG assessment survey; the major part of the assessment is based on a traditional final written exam with closed books. Also the construction of these exams is similar across Europe, possibly around the world. They consist of a number of problems more or less similar to the problems in the textbooks, each given a certain maximum score and together covering most of the intended learning outcomes. When marking an exam the examiner gives the student a score for each problem depending on how successful the student’s attempt to solve the problem turned out to be. The examiner then decides whether the student should pass or fail or get a better grade. Traditionally this decision is entirely depending on the student’s total score. The limit between fail and pass is often set to a percentage of the maximum score. This percentage varies from 40 to 50 or 60. (This has not been investigated).The grading systems vary from country to country, sometimes between universities in the same country. When the ETCS grading system is adopted across Europe it will be of interest to investigate the equality of the grading. But even then the differences between the course modules at different universities will make the comparison very problematic. The aim of this chapter is to discuss the requirements for passing related to the expected learning outcome.

Expected learning outcomes (ELOs) specify depth and what students should be able to do at the end of the module. Typical ELO statements begin `On successful completion of this modules students will be able to' followed by a verb like calculate, solve, explain or prove. The information in the list of ELOs for a course module is twofold. Firstly it tells the student: ‘if you can do all this on the day of the exam you will pass’. Secondly it tells everybody else; ‘a student who passed this course module was able to do all this on the day of the exam’. Or does it really?

This is the examiners dilemma; if the ELOs are expressed in a vague manner they are of little help to the student but one can easily claim equivalence between ‘passing the exam’ and ‘being able to do all this'. For instance an ELO of the type 'after this module the student will be able to solve standard problems in this field of mathematics' gives no information at all to the students but anyone who has passed the exam has certainly managed to do something related to the ELO. On the other hand, if the ELOs are expressed in a precise manner, they are of great help for the students but the equivalence between 'passing the exam' and 'being able to do all this' becomes hard to establish.

To improve quality in teaching and learning we are, according to the constructive alignment principle, supposed to state the ELOs in such a way that they supply the students with a proper guidance for their learning. But the alignment should not be only within the courses but also between courses that together form a program. Furthermore, an improved quality in the education is not achieved automatically just by applying constructive alignment thoughts. It is of course heavily dependent on what the students actually learn. Thus, there is a strong argument for aligning the ELOs and the requirements for passing.

The prevailing principle: 'A student who is given a certain percentage of all possible points on an exam will pass' can no longer be applied, unless the percentage is close to 100. Instead, the requirement for passing should be that the student has demonstrated an acceptable level for all ELOs. There are different ways to design an exam that supports this principle. One way is to have a first part of the exam that only covers methods and procedures which can be rather complicated but standardized and theoretical questions which require a limited understanding and a second part covering problem solving and a higher level of understanding. To pass the student must score a large portion of the first part. The second part is used only for grades above 'passed'. Another design consists of a number of items where the student can show both low and high level of understanding or capability to apply either standard techniques or genuine problem solving in the same field. The criterion for pass is then to score reasonably good on all items. Other exam designs and criteria for pass can serve for the same purpose.

5.5 Examples from exams

6 Conclusions and future developments

Future activities include:

- identifying profiles;
- find further example tasks in order to improve understanding of the competency concept,
- work more on the dimensions in order to see their value;
- work on assessment for competencies;
- find out more about risks and chances of technology use;
- make use of progress in educational research on math didactics;
- perform more studies on workplace mathematics in order to find out about the mathematical challenges engineers meet at their workplace.

For undertaking such efforts, it is important that mathematics lecturers have a long term interest in teaching mathematics to engineers, so the latter should not be a short term duty.

7 Glossary

Attitude

Assessment

Competence

Competency

Connections cluster

Degree of coverage

Profile

Reflection cluster

Reproduction cluster

Technical level

Technology

Workplace mathematics

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